The Canadian Underground Economy: An Examination of Giles and Tedds

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PRÉCIS
Dans leur ouvrage Taxes and the Canadian Underground Economy, David E.A Giles et Lindsay M. Tedds décrivent l'existence d'une importante économie souterraine au Canada qui connaît une croissance rapide. Ils estiment qu'elle équivaut à un montant de revenu additionnel — qui échappe aux administrations et qui n'est pas imposé — variant entre 3,46 % du PIB officiel en 1976 et 15,64 % du PIB en 1995. D'autres personnes ont contesté ces chiffres, mais la méthode économétrique utilisée par Giles et Lindsay est complexe. Dans cet article, Trevor Breusch décortique les modèles, les prévisions et les comparaisons, et révèle l'origine des résultats. Il montre que le cheminement chronologique des estimations des auteurs a peu ou pas à voir avec le revenu souterrain exprimé en pourcentage du PIB et que le niveau général de leurs estimations provient de résultats mathématiques.

ABSTRACT
In their book Taxes and the Canadian Underground Economy, David E.A. Giles and Lindsay M. Tedds describe a hidden economy in Canada that is large and growing rapidly. They estimate an amount of additional income — unobserved by the authorities and untaxed — ranging from a low of 3.46 percent of official GDP in 1976 to a high of 15.64 percent of GDP in 1995. Others have questioned these findings, but the econometric method employed by Giles and Tedds is complex. In this paper, Trevor Breusch peels back the separate layers of model fitting, prediction, and benchmarking and reveals the origins of their results. He shows that the time path of their estimates has little or no connection with underground income as a percentage of GDP and that the overall level of their estimates is a result of numerical accidents.

KEYWORDS: UNDERGROUND ECONOMY ■ TAX EVASION ■ ECONOMETRICS ■ MATHEMATICAL MODELS

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INTRODUCTION

The first third of Taxes and the Canadian Underground Economy by David Giles and Lindsay Tedds1 provides a survey of research on the phenomenon of “unrecorded” or “unreported” income, with emphasis on the estimates that have been made of its size in various countries. Particular attention is paid, naturally, to the evidence for Canada. In the central chapters of the book, the authors describe their techniques for estimating the extent of underground economic activity. They construct a time series of values from 1976 to 1995 to represent hidden income in Canada as a percentage of officially measured GDP. This series is used as data in the latter parts of the book, where the authors examine the relationship of their estimate of the underground economy to other economic variables and the consequences for taxation collection and policy.

I am concerned in this paper with the estimation technique and results of Giles and Tedds, as described in chapters 6 and 7. The basis of their method is the MIMIC (multiple indicator, multiple cause) model of Zellner2 and Goldberger,3 first used in a study of underground economies by Frey and Weck-Hannemann.4 Giles5 gives the approach new life by incorporating the modern econometric tools of cointegration and comprehensive diagnostic testing. His time series of estimates for New

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1 David E.A. Giles and Lindsay M. Tedds, Taxes and the Canadian Underground Economy, Canadian Tax Paper no. 106 (Toronto: Canadian Tax Foundation, 2002).
Zealand is analyzed extensively in Giles, Giles and Caragata, and Caragata and Giles. The first application of the method to Canada is an unpublished MA thesis by Tedds; the Giles and Tedds book is an extension and refinement of that earlier work. Several other papers use the estimated underground economy series from the book as data, including Giles, Tedds, and Werkneh, who examine Granger causality between the underground and measured economies, and Tedds and Giles, who compare Canada with New Zealand.

The MIMIC model approach to estimation of the underground economy appears to be growing in popularity. A number of authors are taking up the Giles and Tedds method: Dell’Anno and Schneider construct a series for Italy and report some results for other OECD countries; Dell’Anno et al., compare France, Spain, and Greece; Bajada and Schneider study Australia and other nations in the Asia-Pacific; in


9 Lindsay M. Tedds, “Measuring the Size of the Hidden Economy in Canada” (MA extended essay, University of Victoria, Department of Economics, 1998).


Schneider et al., the focus is India in comparison with 18 other Asian countries; and in Schneider and Klinglmair there are comparisons of dozens of countries around the world. In addition, Tedds provides some updating and fine-tuning of the Canadian estimates from the book.

The idea of the MIMIC model in this context is to represent the underground economy as a latent variable or index, which has causes and effects that are observable but which cannot itself be directly measured. Thus, there are two kinds of observed variables in the analysis, one consisting of “causal” (or exogenous) variables and another of “indicator” (or endogenous) variables, which are connected by a single unobserved index. Values of the index over time are inferred from data on causes and indicators by estimating the statistical model and predicting the index. The fitted index is then interpreted as a time series of the magnitude of the underground economy. Of course, whether the interpretation is convincing will depend on the specification of the variables.

Some writers argue that the distinctions between “unrecorded” and “unreported” income are so important that it is mistaken to attempt a one-dimensional measure of the underground economy. Even if we agree on that label for the single connection between the groups of variables in the model, we still have to establish that the index is measuring income and not some other dimension of economic transactions, such as turnover. Assuming agreement on that count, it remains to be shown that the index represents underground income measured relative to the official level of national income (the interpretation given by Giles and Tedds) rather than the absolute level of hidden income. In the latter case it is further possible to question whether the income is in nominal money value or deflated by a price index. These uncertainties in what is being measured arise because the MIMIC approach is almost entirely statistical, conducted with little or no guidance from economic reasoning.


There are critics of the uses of MIMIC modelling to estimate the size of the underground economy. In response to Frey and Weck-Hannemann’s seminal study, Helberger and Knepel\(^{19}\) show that the results are unstable in the face of minor changes in either the data period or the list of countries studied. They also argue that the lists of causal and indicator variables are unconvincing for the purpose, and that other aspects of the econometric methods are unsatisfactory.

When the Giles and Tedds book was published, the Canadian Tax Journal carried an introductory note from the journal editor and two papers of scholarly criticism by Smith\(^{20}\) and Hill,\(^{21}\) together with a rejoinder by Giles and Tedds.\(^{22}\) These critics express doubts about the methodology used in the book, especially the complexity of the multi-layer estimation strategy. In an echo of the Helberger and Knepel critique, they question the relevance of the causal and indicator variables selected by Giles and Tedds. They also indicate how the results of such economy-wide investigations are of limited use to policy makers. The matters I examine in this paper are more specific to the estimation method of Giles and Tedds and somewhat more technical than the ones canvassed in the reviews by Smith and Hill.

It will be seen in what follows that there is almost no difference between the measure of the underground economy calculated by Giles and Tedds and just one of their causal variables. The time-series properties of that single variable dictate almost all of their substantive findings about change over time. The variable concerned is a component of national income, and it measures the absolute level of economic activity in a sector, not activity in proportion to measured GDP. Further, the variable is measured in nominal Canadian dollars, since it is not adjusted for inflation in the price level. These findings strain the credibility of the interpretation they give their estimates.

The MIMIC model in the Giles and Tedds approach provides a raw index, which gives only the relative movement in their estimated time series from year to year. The analogy is to a price index, which captures the relative proportional movements over time in the quantity of interest, but which is conventionally scaled to 100 in some base period. In MIMIC modelling the scale of the index is similarly arbitrary, but Giles and Tedds seek to interpret the series as tracking the size of the underground economy. To this effect they specify and estimate a currency demand model, from which they produce auxiliary estimates of the phenomenon of interest. They use these other estimates to calibrate or benchmark the raw index from the MIMIC

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model. They interpret the resulting scaled series as representing underground economy income as a percentage of measured GDP.

I show that the currency model used by Giles and Tedds for benchmarking is unidentified—exactly and globally. Whatever values the parameters might take, there are always some other values with different implications but exactly the same observable consequences. No amount of any conceivable data can distinguish among these observationally equivalent structures. This is a logical problem with the form of the model, and it has profound implications for inference. Fatally for the present purpose, the parameters that determine the benchmark are among the ones that are indeterminate in the model.

Because of the identification failure, the benchmark estimate comes from numerical accidents in the estimation process. I show how radically different estimates are obtained when minor changes are made to the starting values of the estimation algorithm or even when two different, but well-established, software packages are employed for estimation of the same model. Similar problems may be expected when even minor variations are made to the model or in the data, such as when variables or observations are added or dropped. The implication is that the overall level of the series estimated by Giles and Tedds is a mirage.23

THE RESULTS AND ESTIMATION STRATEGY

The Giles and Tedds estimate of the underground economy in Canada is a 20-year time series that ranges from a low of 3.46 percent of official GDP in 1976 to a high of 15.64 percent of GDP in 1995, passing through the benchmark value of 9.45 percent in 1986.24 This series is plotted herein in figure 1. The detailed time shape of the series will be important in the next section, where I consider the predictions of the MIMIC model, but for the moment I will concentrate on the two prominent features that are highlighted with arrows.

First is the strong growth indicated by the sloping arrow. The series increases by a multiple of 4.5 times over 20 years, which is equivalent to a compound rate of increase of 7.8 percent per annum. This phenomenal growth is more remarkable for being relative to observed GDP. Over the same period, recorded Canadian real GDP grew by 64 percent. Thus it seems that the level of underground income in Canada increased in real terms more than seven times over while the observed

23 Smith, supra note 20, at 1656, observes in reference to the currency model: “The particular model they use is subject to many of the same criticisms levied against other models—for example, that minor changes may lead to substantially different results.” No further evidence is offered, so this seems to be a general comment about a class of models, or perhaps about models in general, and not a demonstrated defect of this particular model.

24 The time series is listed in Giles and Tedds, supra note 1, at 120, table 7.1.

25 This calculation assumes that the Giles and Tedds index is the relationship of real underground income to real measured GDP, or if the relationship is in nominal variables, that the implicit GDP price deflator applies to underground economy income.
economy much less than doubled in size. To achieve that increase in 20 years requires a compound annual rate of real growth in the underground economy of 10.5 percent. This rate is unrealistic. As will be seen, these growth properties of the underground economy are derived entirely from fitting and predicting the MIMIC model.

The second important feature is the benchmark value of 9.45 percent of GDP for 1986, which is marked with the vertical arrow in figure 1. Whether this number is high or low is a matter for debate, but Giles and Tedds cite other studies that produce estimates of similar magnitude. We will see that the overall magnitude of the underground economy estimate is derived from the currency demand model.

In the MIMIC model the scalar latent variable, $\eta_t$, is written as a linear combination of a $q \times 1$ vector of causal or exogenous variables, $x_t$,

$$\eta_t = \gamma' x_t + \zeta_t \quad t = 1, \ldots, N,$$

where $\gamma$ is a vector of unknown coefficients and $\zeta_t$ is an unobserved error term with zero mean and constant variance. There are other parts to the model involving the indicator variables, but for the present purpose all that matters is the insight that,

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26 Growth in the index is sometimes much higher than average, to the extent that the level of real underground income increases by over 20 percent in some years.
given the estimate of the coefficient vector, the estimated index is obtained as the prediction

$$\hat{\eta}_t = \hat{\gamma} \hat{x}_t, \quad t = 1, \ldots, N. \quad (2)$$

The reason for mentioning an otherwise trivial and obvious point is to flag the point that the prediction formula in (2) is not used by Giles and Tedds. Rather, they obtain the prediction as

$$\hat{\eta}_t = \hat{\gamma} \hat{x}_t^*, \quad t = 1, \ldots, N, \quad (3)$$

where $x_t^*$ is a transformed version of $x_t$.

What is the transformation? To address the obviously trending nature of the data on the causal and indicator variables, and to protect the estimation results against a charge of spurious regression, they difference the data to stationarity before fitting the model. The prediction stage, however, is based on the actual values of the variables (the levels), so that the $x_t^*$ variables in our notation are integrated versions of the $x_t$ variables. Of course, the time series properties of the integrated variables are very different from those of the differenced variables (especially so, since one of the variables is a component of Canadian national income in nominal dollars, and has to be differenced twice to make it stationary).

In equation 1 the dependent variable $\eta_t$ is unobserved, and both $\gamma$ and $\text{var}(\zeta_t)$ are parameters to be determined in estimation. Hence there is nothing in the equation to determine the scale of the unobserved index. The convention in MIMIC modelling is to relate the index to one of the indicator variables. Giles and Tedds follow this convention temporarily, and then reset the scale from the results of the currency demand modelling. From this other model they derive another time series of the underground economy. They use

the average value of approximately 9.45 percent of measured GDP as a representative estimate of the ratio over the sample period. . . .

Since the sample mean value of measured GDP occurs in 1986, we set the ratio of underground GDP to measured GDP to 9.45 percent for that year in order to convert the index time-path for the hidden economy that emerged from our MIMIC-model analysis.27

Thus the index in 1986 has the value 9.45 derived from the currency demand model, as shown by the vertical arrow in figure 1. The rest of the benchmarked series is given by the scaling operation

$$\text{index}_t = 9.45 \times \frac{\hat{\eta}_t}{\hat{\eta}_{1986}} \quad \text{for all } t = 1976, \ldots, 1995. \quad (4)$$

This confirms my earlier observation that the overall magnitude of the underground economy series is derived from the currency demand model.

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27 Supra note 1, at 138.
A simple but important property of the benchmarking process is its preservation of the relativities between the measurements in different years, so that

\[
\frac{\text{index}_t}{\text{index}_s} = \frac{\hat{h}_t}{\hat{h}_s} \quad \text{for all } t \text{ and } s.
\]  

(5)

This confirms the earlier observation that the proportional growth properties of the scaled index are derived entirely from fitting and predicting the MIMIC model. I represent this feature by the sloping arrow in figure 1, showing the growth of the (scaled) index between 1976 and 1995.

I am able to replicate the results of Giles and Tedds using the estimates of the coefficient vector (\(\hat{\gamma}\)) as published and the data on the causal variables (the components of \(x_t^\star\)) supplied to me by Lindsay Tedds. With these inputs in formulas 3 and 4, I get a series that is almost identical to the published index.

**MIMIC MODEL PREDICTION**

What is the effect of predicting from the MIMIC model with the data in levels when the model has been estimated with the data in differences (in one case differenced twice)? Again, I use the \(\hat{\gamma}\) coefficient estimates as published, shown as the first row of table 1, labelled “Gamma.” In the next three rows of the table are some summary statistics of the causal variables, \(x_t^\star\). The two rows with obvious labels contain the mean and standard deviation of the variable over the 20 years. The row labelled “Growth” is the ratio of the variable in 1995 to its value in 1976, thus showing the growth multiple in 20 years.

For the moment it is best to set aside the meaning of the variables (details can be found in table A1) and just focus on construction of the raw index as a prediction. A feature of the summary statistics is the enormous difference in scale between the first three variables (particularly the first two) and the remaining four. Whether scale is measured in means or standard deviations, these variables are of very different orders of magnitude. Any linear combination with similar-sized coefficients will be dominated almost exclusively by the first two or three variables. A second

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28 Coefficient estimates of the “preferred model” are given as model 6 in Giles and Tedds, ibid., at 130, table 7.4, and shown in the first row of table 1 herein. The data as supplied are listed in the appendix, together with the names of the relevant variable from Giles and Tedds, ibid., at 129, table 7.2.

29 In my replication of the scaled index, I get a range from 3.45 to 15.71, compared to the range 3.46 to 15.64 in the published results. This divergence is small enough to be explicable by my use of the published coefficients of the MIMIC model, which are rounded to three significant digits. While I can replicate the index using the published coefficients and the supplied data, I have been unable to replicate estimation of the MIMIC coefficients; hence my use of the published coefficients throughout.

30 Indeed, the published coefficients not only prove impossible for me to replicate, on this evidence they seem implausible. The variables in the model (even after differencing) are enormously different in scale—their means and standard deviations differ by factors in the millions—but the coefficients are all nicely in the range 0.028 to 0.627.
A feature of note is that only the first two variables have growth factors anything like the value of 4.5 found in the Giles and Tedds estimate of the underground economy in Canada. Even then, the factor of 3.17 in MULT falls well short, so we deduce that SELF has to be an important driver in the index series.

We can use the analogy with a price index to assess the relative importance of the variables in forming the prediction from the MIMIC model. The predictor in expression 3 above is equivalent to a Laspeyres price index, with a price vector in each period of \( x_t^* \) and base period quantities in the vector \( \hat{x}_{j0} \). Scaled in the conventional way to 100 in the base period (indicated by \( t = 0 \)), the index is

\[
L_t = 100 \times \left( \frac{\sum_{j=1}^{7} \hat{y}_j x_{jt}^*}{\sum_{j=1}^{7} \hat{y}_j x_{j0}^*} \right) \quad \text{for} \ t = 1976, \ldots, 1995. \tag{6}
\]

Then, using the usual tool kit from index number theory, we can write this expression as a weighted combination,

\[
L_t = 100 \times \sum_{j=1}^{7} \left( \frac{x_{jt}^*}{x_{j0}^*} \right) S_{jt} \quad \text{for} \ t = 1976, \ldots, 1995, \tag{7}
\]

where the relative “prices” over time are the growth factors \( x_{jt}^*/x_{j0}^* \) and the weights are the base-period “budget shares,”

\[
S_{kt} = \frac{\hat{y}_k x_{j0}^*}{\left( \sum_{j=1}^{7} \hat{y}_j x_{j0}^* \right)} \quad \text{for} \ k = 1, \ldots, 7. \tag{8}
\]

Thus the row of table 1 labelled “Growth” can also be interpreted as the growth ratios of the seven individual variables that are to be combined into a single index, measured from a base in 1976 to the end of the period in 1995. The corresponding

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**Table 1** Selected Properties of the “Causal” Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>MULT</th>
<th>SELF</th>
<th>INC</th>
<th>ERTE</th>
<th>CORP</th>
<th>IND</th>
<th>UNEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma</td>
<td>0.627</td>
<td>0.359</td>
<td>0.028</td>
<td>0.293</td>
<td>0.268</td>
<td>0.224</td>
<td>0.308</td>
</tr>
<tr>
<td>Mean</td>
<td>4.66E+05</td>
<td>9.68E+06</td>
<td>2.55E+04</td>
<td>1.23</td>
<td>2.99</td>
<td>12.86</td>
<td>9.29</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>1.50E+05</td>
<td>4.56E+06</td>
<td>4.34E+02</td>
<td>0.10</td>
<td>0.53</td>
<td>0.75</td>
<td>1.56</td>
</tr>
<tr>
<td>Growth</td>
<td>3.17</td>
<td>4.71</td>
<td>1.00</td>
<td>1.39</td>
<td>0.72</td>
<td>1.09</td>
<td>1.32</td>
</tr>
<tr>
<td>Share</td>
<td>9.42%</td>
<td>90.53%</td>
<td>0.05%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Source: David E.A. Giles and Lindsay M. Tedds, *Taxes and the Canadian Underground Economy*, Canadian Tax Paper no. 106 (Toronto: Canadian Tax Foundation, 2002), 120, table 7.1 and 130, table 7.4; and table A2 of this article.

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31 Luckily, the variables and their coefficients are always positive; this need not be so in a MIMIC model, but it is needed in the analogy with a price index.
base-period weights or budget shares are shown in the last row of table 1, labelled “Share.” Here we see that more that 90 percent of the movement in the index can be attributed to the variable SELF. Of what remains, almost all is due to MULT, and these two variables together account for 99.95 percent of the index. The very small amount not explained by SELF and MULT is almost entirely due to the third variable, INC. The contribution from variables four through seven in the list is effectively zero.

Before asking what these variables measure and how that affects the interpretation given to the index by Giles and Tedds, there is another way to show the dominance of SELF in forming the index. Suppose SELF alone is the raw index, and consider what happens if this variable is benchmarked in the same way as the prediction from the MIMIC model and then presented as the estimate of the underground economy. The simple correlation between SELF and the index of Giles and Tedds is 0.9999. It might be argued that this correlation is inflated by the trends in both variables, but the correlation between the first differences is also very high at 0.9922. Even the second differences series, which in the case of SELF is stationary according to Giles and Tedds, have a correlation coefficient of 0.9858.

Comparisons by correlation may overstate the similarity between the variables as descriptions of a time path. Correlations are invariant to changes in both location and scale in the variables, but in forming an index series only the scale is freely adjusted. A better time-series comparison for this purpose can be seen in the plot points of figure 2. Here SELF has been scaled to the benchmark value of 9.45 in 1986, in exactly the same treatment as the scaled index. It is again obvious that the two series are almost identical. The main distinction is the slightly flatter overall slope of the index compared to SELF; this is mostly due to the minor contribution made to the index by MULT. Further, the plot points of the first differences of the two variables nearly coincide, except the index is slightly below SELF on average. Even the second differences are almost coincident throughout the period. It is clear that SELF drives almost all of the behaviour of the index apart from the level (which is set by the scaling).

What are the variables that constitute the index of Giles and Tedds? For convenience, descriptions of the relevant variables are given in table A1.32 The dominant influence, SELF, is described as “Income of self-employed persons.” As noted by Hill,33 this variable is an economy-wide aggregate and measured in nominal dollars. SELF increases by a multiple of 4.71 over the 20 years, and we have seen that this variable drives the growth in the index almost on its own. Over the same period, nominal GDP in Canada rises by a multiple of 3.92, so that the income of the self-employed grows only slightly in relation to GDP, by a multiple of $4.71/3.92 = 1.20$. The growth multiple in nominal GDP is composed of a component factor in real GDP of 1.64 and one in prices of 2.39. Expressed as changes, the income of self-employed

32 Extracted from Giles and Tedds, supra note 1, at 129, table 7.2.
33 Supra note 21.
FIGURE 2  Scaled Index and SELF

Levels

First differences

Second differences

--- Scaled index
--- Scaled SELF

Source: David E.A. Giles and Lindsay M. Tedds, *Taxes and the Canadian Underground Economy*, Canadian Tax Paper no. 106 (Toronto: Canadian Tax Foundation, 2002), 120, table 7.1; and table A2 of this article.
persons rises by 20 percent relative to income generally, real national income goes up by 64 percent, and price inflation accounts for the remaining 139 percent of the recorded growth in SELF. Thus, most of the high level of growth in this dominant “cause” of the underground economy comes from price inflation, a further substantial part is movement in concert with the general expansion of the Canadian economy as measured by real GDP, and only a small part is due to the contributions of the self-employed relative to the economy as a whole. It is quite implausible that it charts the growth of the underground economy as a percentage of measured GDP.

The only other variable of note in the index is MULT, described as “The total number of male holders of multiple jobs aged 15 or more.” Again, this is another national aggregate, not a proportional factor in measured GDP, although this time it is a real variable and not one in nominal dollars. Hill describes the selection of the causal and indicator variables in the MIMIC model as “somewhat ad hoc,” an assessment I would endorse were it not for the understatement. I can add the rider that the only variables making a significant contribution to the index of Giles and Tedds are ones that, in measurement terms, are most inappropriate. Other causal variables in the model are more plausible as drivers of an underground economy (for instance, CORP and IND, which are ratios of taxes to GDP), but they are irrelevant to the calculation of the index.

This finding that SELF alone determines most of the time profile of the index, with just a little help from MULT, neatly explains the “robustness” that Giles and Tedds promote as a virtue:

We considered about a dozen different MIMIC-model specifications, and we found that the results were extremely “robust”; that is to say, the time-paths for the underground economy yielded as “predictions” by the models were all closely similar—for the most part, indeed, virtually indistinguishable.

The one feature common to all 10 different MIMIC models reported in their table 7.4 is inclusion of SELF and MULT as causal variables.

I note here that Giles and Tedds are wrong in describing the relative importance of individual variables (a description echoed by Hill in his comments on their specification). They say,

the weight for each of the other variables indicates its importance as an indicator or a cause. . . . Self-employment is somewhat more important than either the exchange rate or unemployment. Corporate and indirect taxes together account for about 23 percent of the overall weight of the causes. The holding of more than one job is a particularly important factor; it accounts for nearly 30 percent of the total weight.

34 Ibid., at 1644.

35 Supra note 1, at 115 (quotation marks in original).

36 Ibid., at 116.
From the numbers being quoted, it is clear that their calculation of a weight is the individual coefficient in proportion to the sum of the coefficients. Thus, in contrast to the shares described in 8 above, their interpretation of the weights is

$$W_k = \frac{\hat{r}_k}{\sum_{j=1}^{7} \hat{r}_j} \quad \text{for} \quad k = 1, \ldots, 7.$$  

(9)

This interpretation does not make any sense. In a MIMIC model, as in simple linear regression, the magnitude of a coefficient is a function of the units in which its variable is measured. When a variable $x_{jt}$ is rescaled one way or another, its coefficient $\hat{r}_j$ is scaled by precisely the inverse transformation. Hence a coefficient can be made arbitrarily large or small by transforming the corresponding variable to suit. Simply quoting the relative size of a coefficient says nothing about the relative importance of that variable in a linear combination—unless the data are transformed to make all the variables essentially equal.

Perhaps some further transformations are applied to the data before the MIMIC model is estimated to obtain the published coefficients (in addition to the differencing that is documented). One possibility is that the variables are standardized by transforming them individually to zero mean and unit standard deviation. Such additional transformations are not only undocumented, they are another wedge between the $x_{jt}$ variables used in estimation and the $x_{jt}^*$ variables used in prediction. In particular, if the estimation variables are standardized, the estimated coefficients will be invariant to the units of measurement in the original variables. But when the index is formed by applying these coefficients to the original variables—as I have shown by replication is the procedure—the calculation will be quite sensitive to the units in which the variables are measured. Different substantive answers will be obtained, for instance, if SELF is measured in units of millions of dollars (as GDP is measured in the data) rather than what seems to be thousands of dollars. But this speculation about the estimation procedure does not change the properties of the index. The calculation of the relative importance of each variable to the index must be made from the actual coefficients and variables that are used in prediction, as I have done.37

**CURRENCY DEMAND MODEL**

While the time shape of the estimates of the underground economy comes from predicting the MIMIC model, the overall level of the estimates is set by benchmarking from a separate currency demand model. Giles and Tedds specify the estimation model for currency as follows.38 Demand for (nominal) currency is written as

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37 Standardization might explain the puzzling fact that the variables are so different in magnitude while the reported coefficients are all nicely within a small range. Indeed, my replication attempts are much closer to the reported results when all the variables in the estimation model are standardized. Nonetheless, the reported results cannot be replicated, which suggests that there may be some other undocumented data transformation.

38 Giles and Tedds, supra note 1, at 109-10 and 134-36.
where $Y_{Rt}$ and $Y_{Ht}$ are “recorded” and “hidden” real income (or output), respectively; $R_t$ is the interest rate; $P_t$ is the price level; and $e_t$ is an error term. Some assumptions are required to represent hidden income $Y_{Ht}$ as a function of observable variables. After considering a longer distributed lag relationship between hidden and recorded income, and setting some interim lag coefficients to zero based on initial testing, they arrive at the relationship

$$Y_{Ht} = a_0 Y_{Rt} + a_2 Y_{Rt-2}.$$  

Substituting 10 into 6.4, and linearizing by taking logarithms, yields the “preferred” form of the model

$$\log M_t = \log b_0 + b_1 \log (Y_{Rt} + a_0 Y_{Rt} + a_2 Y_{Rt-2}) + b_2 \log R_t + b_3 \log P_t + v_t,$$

where $v_t = \log e_t$ is $NID(0, \sigma^2)$. The model they estimate is an “improved” specification, containing an additional lag of the (log) interest rate; thus

$$m_t = b_0^* + b_1 \log [(1 + a_0) Y_{Rt} + a_2 Y_{Rt-2}] + b_2 r_t + b_3 p_t + b_4 r_{t-1} + v_t,$$

where $b_0^* = \log b_0$ (I have adopted the shorthand notation of lower-case letters for logarithms of variables). Estimates by maximum likelihood (equivalent to nonlinear least squares) are reported in Giles and Tedds. For convenient reference, these same results are listed in column 1 of my table 2. With a minor exception to be noted below, I have been able to replicate these results exactly using the data in the appendix.

Giles and Tedds discuss some antecedents in the literature of this form of currency demand model, but they give no indication of expected values of the parameters. I note that $b_0 > 0$ is required for the log representation of the intercept, but beyond that nothing much can be said about that parameter. The exponents in 6.4 are elasticities of currency demand with respect to real income, the interest rate, and the price level, so that we can expect $b_1 > 0$, $b_2 < 0$, and $b_3 > 0$ on the usual
reasoning for money demand models. Homogeneity of money demand in nominal income at a constant interest rate suggests $b_1 = b_3 = 1$ might hold or at least might be an interesting hypothesis to explore. In the estimation model 7.8, the interest rate elasticity is $b_2$ in the short run and $b_2 + b_4$ in the long run. It seems likely that both $b_2 < 0$ and $b_4 < 0$, although these are fairly weak predictions compared with the stronger one that $b_2 + b_4 < 0$.

Parameters $a_0$ and $a_2$ represent the current-period multiplier and the two-quarter-lag multiplier, respectively, of hidden income in response to recorded income. It is unclear what signs these parameters might have individually, because there is no explanation given for the relationship. There may be substitutions between sectors as well as co-movement of the two sectors, and the behaviours may be different in the short run and the long run. If the recorded part of the real economy is static, so that $Y_{Rt} = Y_{Rt-2}$, then the ratio of hidden to recorded income is $a_0 + a_2$, which is constant and must be non-negative. On the other hand, if the recorded real economy is growing steadily at a rate of $g$ per annum, the ratio of hidden to recorded income is 

$$
\frac{Y_{Ht}}{Y_{Rt}} = \left[ a_0 + a_2 (1 + g)^{-2} \right],
$$

(11)

which is again constant. In this case, the non-negativity requirement on hidden income imposes a restriction on $a_0$ and $a_2$ that involves the growth rate.

With these prior requirements on the signs of the parameters in 7.8, we see that the reported estimates are mostly quite plausible. The $b_0$ estimate reported in Giles and Tedds is negative, which presents a problem because the intercept is written as $\log b_0$. However, from my replication of the results it is seen that the quantity reported is not $b_0$ but rather $b_0 = \log b_0$. The signs of the estimates $b_1$ through $b_4$ are all as predicted, although $b_1$ may be larger in magnitude, and $b_3$ smaller, than would follow from the homogeneity hypothesis. The estimates of $a_0$ and $a_2$ imply a ratio of hidden to recorded income in a static economy of 12.6 percent. If the growth rate in real income is assumed to be a steady 2.6 percent per annum (the average for Canada over the sample period) then the ratio of hidden to recorded income is indicated to be 8.8 percent.

With these estimated parameters from the currency demand equation and the actual sample values of recorded real income from 1974 onward as listed in the appendix, we can calculate the time path of hidden income for 1976-1995. From equation 10 we have $\hat{Y}_{Ht} = \hat{a}_0 Y_{Rt} + \hat{a}_2 Y_{Rt-2}$, where $\hat{a}_0 = -0.4618$ and $\hat{a}_2 = 0.5877$. When the calculation is converted to a percentage of recorded GDP, the original benchmarking results are replicated almost exactly.42

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42 I agree with the reported maximum of 13.80 percent and find an average of 9.459 (which is closer to 9.46 than to 9.45). However, the smallest value in the period is 6.59 percent (which occurs in 1985), not 7.25 percent as reported, which is the second smallest. The time profile looks nothing like the series from the MIMIC model.
Identification in the Currency Demand Model

The replication of benchmarking takes the published estimates of the currency demand model as given. Unfortunately, as I will now show, these “estimates” have little or nothing to do with any evidence in the data. The problem is that the model is not identified, and the parameters used for estimating the underground economy are among those that are indeterminate.

Identification is the requirement that distinct configurations of parameter values can be distinguished empirically, at least in principle, if the right data can be found. Statistically this requires the likelihood function to be different throughout the parameter space. If some value of the parameters is unidentified, then there are other values of the parameters (often infinitely many other values) with different meanings in the model but with exactly the same likelihood function. With the same likelihood function, the mapping from data to parameters that supports one configuration of parameters will equally support others. No amount of any conceivable data can distinguish between these observationally equivalent structures in an unidentified model.

While identification is a logical property of the model, absence of identification has practical consequences for estimation. The most severe will be a failure to produce estimates, because the algorithm for likelihood maximization finds many, equally good, directions to proceed. Different software packages will render the failure differently. Some may simply crash when the going gets tough. Others may be unstable and produce estimates or standard errors that depend on the starting values, perhaps without any warning that termination has been premature. More sophisticated software may be programmed to drop parameters to resolve the ambiguity and to report estimates based on a subset of the parameters. Since there are no established rules for this treatment, different packages may produce different results.

I want to explore the identification of parameters in the currency demand model of Giles and Tedds—without investing in the apparatus of a likelihood function. To this end, the model can be rearranged and the identification question considered in a slightly different parametric form. The first step is to use the properties of logarithms to write equation 7.8 as follows

\[ m_t = b_0^* + b_1 \log d_1 + b_1 \log(Y_{rt} + d_2 Y_{rt-2}) + b_2 r_t + b_3 p_t + b_4 r_{t-1} + \nu_t, \]  

where \( d_1 = (1 + a_0) \) and \( d_2 = a_2 / (1 + a_0) \). Note that the model has not been changed in any material way: it has just been rearranged and some of its components have been relabelled. Importantly, knowing \((a_0, a_2)\) is as good as knowing \((d_1, d_2)\), and vice versa.43

Consider equation 12 as if it were a slightly complicated linear regression model. The last three terms will serve to identify their own coefficients \(b_2, b_3,\) and \(b_4\) in the

---

43 The statement is not quite true if \(a_0 = -1\) and \(a_2 = 0\), but in that case the interpretation of underground economy is nonsense.
usual way. Then the first three terms are required to identify $b_0^*, b_1, d_1$, and $d_2$. It is reasonable to expect that $b_1$ and $d_2$ can be extracted separately from the nonlinear third term, provided there is enough independent variation over time in real income to ensure that $Y_{Rt}$ is clearly distinguishable from $Y_{Rt-2}$. But the first two terms are just components of the intercept in this model, which has the composite form $b_0^* + b_1 \log d_1$. Even if one is armed with the value of $b_1$ from the previous step, it is impossible to solve this composite intercept for the two separate parameters $b_0^*$ and $d_1$, which indicates that these parameters cannot be separately identified. In terms of the original parameters, since $d_1$ is unidentified so are both $a_0$ and $a_2$. Thus, the parameters essential to forming the benchmark estimate of the underground economy are indeterminate.

**An Empirical Demonstration**

We expect econometric software to experience problems with this currency demand model, perhaps failing to produce estimates or implementing some mechanical rule to remove the identification problem and thereby enable estimation of the other parameters. If estimates are produced, we expect them to be sensitive to features of the method that should be irrelevant, such as the particular estimation algorithm or the starting values used in the algorithm. In particular, the parameters used to infer the benchmark level of the underground economy will be among the least reliable.

The estimates reported by Giles and Tedds could no doubt be replicated with knowledge of the particular estimation algorithm and starting values that they used. However, the more interesting issue is the problems caused by identification failure in this model. To illustrate empirically, I will employ two software packages and make small perturbations to the starting values. One package is Shazam version 8.0 of 1997, because that was used by Giles and Tedds. I also use Stata version 8.2 of 2004 as an example of a modern, highly regarded, econometrics package. In both of these programs the command for nonlinear regression estimation is called “NL.” All options on these commands are set at their defaults.

I find markedly different results from those published. In addition, the results obtained from the two packages vary wildly from each other and show further instability when different initial starting values are provided. The results are recorded in table 2, where column 1 reports the published results. Columns 2 through 5 contain my estimates by Shazam, while columns 6 through 9 contain estimates from Stata. For each package there are four sets of estimates, distinguished by the different starting values given to the algorithm. Below each coefficient estimate is its t-ratio as reported by the package. To show the overall effects of the different experiments, across the bottom of table 2 are recorded the number of iterations taken by the nonlinear algorithm, the residual sum of squares at termination (which determines

44 The Shazam 8.0 manual is referenced in Giles and Tedds, supra note 1, at 141, note 30.
45 Taken from Giles and Tedds, ibid., at 138, table 7.7.
### TABLE 2 Re-Estimates of Currency Demand Model

<table>
<thead>
<tr>
<th>Coef (t-ratio)</th>
<th>(1) Published Shazam</th>
<th>(2) Shazam start 0</th>
<th>(3) Shazam start 1</th>
<th>(4) Shazam start 2</th>
<th>(5) Shazam start 3</th>
<th>(6) Stata start 0</th>
<th>(7) Stata start 1</th>
<th>(8) Stata start 2</th>
<th>(9) Stata start 3</th>
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</thead>
<tbody>
<tr>
<td>$b_0$</td>
<td>-0.3702 (0.31)</td>
<td>-0.3702 (0.40)</td>
<td>-0.5316 (0.38)</td>
<td>-0.3749 (0.29)</td>
<td>-0.2971 (0.21)</td>
<td>-0.3702 (***</td>
<td>-0.3702 (***</td>
<td>-0.2231 (***</td>
<td>-0.6931 (***</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1.6587 (9.07)</td>
<td>1.6587 (9.08)</td>
<td>1.6587 (9.10)</td>
<td>1.6587 (9.16)</td>
<td>1.6587 (7.60)</td>
<td>1.6587 (7.60)</td>
<td>1.6587 (7.60)</td>
<td>1.6587 (7.60)</td>
<td>1.6587 (7.60)</td>
</tr>
<tr>
<td>$b_2$</td>
<td>-0.0751 (2.71)</td>
<td>-0.0751 (2.75)</td>
<td>-0.0751 (2.73)</td>
<td>-0.0751 (2.75)</td>
<td>-0.0751 (2.74)</td>
<td>-0.0751 (2.31)</td>
<td>-0.0751 (2.31)</td>
<td>-0.0751 (2.31)</td>
<td>-0.0751 (2.31)</td>
</tr>
<tr>
<td>$b_3$</td>
<td>0.4041 (3.90)</td>
<td>0.4041 (3.91)</td>
<td>0.4041 (3.93)</td>
<td>0.4041 (3.95)</td>
<td>0.4041 (3.27)</td>
<td>0.4041 (3.27)</td>
<td>0.4041 (3.27)</td>
<td>0.4041 (3.27)</td>
<td>0.4041 (3.27)</td>
</tr>
<tr>
<td>$b_4$</td>
<td>-0.1148 (3.64)</td>
<td>-0.1148 (3.65)</td>
<td>-0.1148 (3.65)</td>
<td>-0.1148 (3.66)</td>
<td>-0.1148 (3.66)</td>
<td>-0.1148 (3.07)</td>
<td>-0.1148 (3.07)</td>
<td>-0.1148 (3.07)</td>
<td>-0.1148 (3.07)</td>
</tr>
<tr>
<td>$a_0$</td>
<td>-0.4618 (3.37)</td>
<td>-0.4618 (1.04)</td>
<td>-0.4068 (2.34)</td>
<td>-0.4603 (3.07)</td>
<td>-0.4850 (3.11)</td>
<td>-0.4618 (0.97)</td>
<td>-0.4618 (0.97)</td>
<td>-0.5074 (1.18)</td>
<td>-0.3461 (0.58)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.5877 (4.26)</td>
<td>0.5877 (1.20)</td>
<td>0.6477 (3.79)</td>
<td>0.5893 (3.64)</td>
<td>0.5623 (3.35)</td>
<td>0.5877 (1.11)</td>
<td>0.5877 (1.12)</td>
<td>0.5378 (1.12)</td>
<td>0.7140 (1.08)</td>
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<tr>
<td><strong>Iterations</strong></td>
<td>9</td>
<td>26</td>
<td>17</td>
<td>15</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td><strong>Residual SS</strong></td>
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<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
<td>0.014942</td>
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<tr>
<td><strong>Benchmark, %</strong></td>
<td>9.46</td>
<td>20.6</td>
<td>9.77</td>
<td>4.74</td>
<td>9.46</td>
<td>9.46</td>
<td>0.17</td>
<td>33.0</td>
<td></td>
</tr>
</tbody>
</table>

---

**Starting values:**
- Start 0 = estimates as published.
- Start 1 = $b_0^*$ as published, $b_1 = b_3 = 1.0, b_2 = b_4 = -0.1, a_0 = -0.5, a_2 = 0.6$.
- Start 2 = as published, except $b_0^*$ varied to log(0.8) = -0.2231.
- Start 3 = as published, except $b_0^*$ varied to log(0.5) = -0.6931.

**SS = sum of squares.**
the value of the likelihood function), and the benchmark value for the underground economy obtained from estimates of \( a_0 \) and \( a_2 \) by the method described previously.

To check the setup—and let the reported results make their best showing—I begin with the published estimates as the starting values. Hence the initial values fed to both Shazam and Stata for this first exercise are \(-0.3702\) for \( b_0^* \) and \(1.6587\) for \( b_1 \), etc. These results are reported under “Start 0” in columns 2 and 6 of table 2. Both Shazam and Stata replicate all of the coefficients very closely, confirming that the reported estimates are a stationary point of the likelihood function. Shazam takes nine iterations to converge while Stata requires only one, indicating the comparative efficiency of the two algorithms. Looking at the t-ratios for \( b_1 \) through \( b_4 \), we see that Shazam is uniformly more optimistic than Stata about significance of estimates, presumably reflecting the different ways that standard errors can be validly obtained in nonlinear estimation. Since the estimates of \( a_0 \) and \( a_2 \) are the same in both cases, and the same as the initial values, the calculation of the benchmark is 9.46 percent, as it is from the published estimates.

One remarkable difference is the t-ratios on the key parameters \( a_0 \) and \( a_2 \). Although Shazam is again slightly more optimistic than Stata about significance, in neither package is either of these two parameters remotely significantly different from zero at the usual levels. This contrasts strongly with the t-values stated in the published results of 3.37 and 4.26, respectively, which the authors find “encouraging.”46 I cannot account for this discrepancy, except as an example of the wild variability of estimation results in an unidentified model.

The other contrast in the estimates in columns 2 and 6 is the t-ratio on \( b_0^* \). Both Shazam and Stata return the starting value as the coefficient estimate, and Shazam gives a standard error and therefore a t-ratio to go with the coefficient. But Stata does not provide a standard error, and hence there is no t-ratio on this coefficient. In this model it is theoretically impossible to estimate all three of \( a_0 \), \( a_2 \), and \( b_0^* \) without some extraneous information. Stata’s response, quite sensibly, is to fix \( b_0^* \) at its initial value and to estimate the other two. Shazam somehow pretends that all three parameters are estimated and quotes a standard error for each of them.

Next we look at the effect of perturbing the starting values slightly. First, \( b_0^* \) is kept fixed at the published estimate of \(-0.3702\) and all the others changed to values in the plausible range not too far from the published estimates. These results are reported under “Start 1” in columns 3 and 7. In this case, Stata robustly converges to the same estimates as before, taking just three iterations to do so. Shazam, however, is upset by the change in initial values and produces a completely different set of estimates for the three unidentified parameters. These new estimates are equally valid because the likelihood value is equally maximized at this point. But now the implications for the benchmark are completely different: the new coefficients imply a benchmark estimate of the underground economy at 20.6 percent of GDP. This time the t-ratios reported by Shazam on the important estimates are indeed “encouraging.”

46 Ibid., at 137.
The other four columns in Table 2 record experiments where only $b_0^*$ is varied from its starting value of $-0.3702 = \log(0.7)$, and all the other parameters are started from their published estimates. Under “Start 2” in columns 4 and 8, the variation is up slightly to $\log(0.8) = -0.2231$, and under “Start 3” in columns 5 and 9 the variation is down slightly to $\log(0.5) = -0.6931$. As with the previous perturbation of the starting values, Shazam needs more iterations to achieve convergence than before and many more than Stata. Stata recognizes the lack of identification, holds $b_0^*$ at its initial value, and estimates the other parameters conditional on that value. Shazam, by contrast, presents estimates for all of the parameters. But the arbitrariness of the “estimation” solution in all cases is evident from the variation in the estimates of $a_0$ and $a_2$ across the experiments. The implied benchmark of the underground economy in Canada varies from 4.74 to 20.6 percent of GDP in the estimates from Shazam, and from 0.17 percent to 33.0 percent in the estimates from Stata. Even more dramatic variation can be achieved with larger perturbations of the initial conditions. In this light, it is difficult to accept the authors’ assurance that “[t]he results in the table are robust with respect to the choice of starting values for the non-linear estimation algorithm.”

CONCLUSIONS

Giles and Tedds estimate an underground economy in Canada that is remarkably large and growing rapidly. Their time series of hidden economic activity as a percentage of measured GDP rises from 3.46 percent in 1976 to 15.64 percent in 1995. However, I can account for their findings by features of the modelling and data that have little or nothing to do with their interpretation of the series.

Understanding their results requires peeling away the layers of econometric complication, which include MIMIC modelling, prediction, and benchmarking. I am able to show that the overall time pattern of their series, including its surprising growth by a factor of 4.5 over 20 years, is attributable to their prediction formula. Their index of underground economic activity is derived almost entirely from a single variable—namely, the declared incomes of self-employed persons. This variable is an economy-wide aggregate and measured in nominal dollars. Naturally, it experiences strong growth from the mid-1970s to the mid-1990s, in part due to real growth of the Canadian economy but mostly due to price inflation. There is nothing in such a variable to sustain its interpretation as an index of underground economic activity relative to observed GDP.

The overall level of the index is established by a benchmark from a separate currency demand model. This model is unidentified, so the “estimates” of its key parameters are merely numerical accidents without connection to the data. I show

47 Ibid. My findings are more consistent with this revised view of one of the authors: “This model, however, has subsequently been found to be extremely unstable. In particular, the highly nonlinear model that was used in the previous work proved to be sensitive to small data changes.” Tedds, supra note 17, at 11 and note 8.
that a wide variety of benchmark values can be obtained by tweaking the estimation method in ways that should have no effect.

This leaves the question of why the numbers have any plausibility at all: why not an estimate of the underground economy in Canada that is negative or larger than 100 percent? The answer must lie in the control that researchers exercise over their methods to ensure that the results are interesting and reasonable (meaning challenging but not too outlandish). Quite rightly, Giles and Tedds are careful to place their results in the context of the literature, particularly the existing estimates for Canada. They cite the similarity of their results to other findings (at least on average) as validating their methodology, while promoting the novelty of their 20-year time path. But it seems that their published results are chosen from among a wide spectrum of equally valid possibilities, not just by reference to the evidence in their data and the internal logic of their modelling, but also on an understanding of what numbers are interesting and reasonable.

While this paper has been limited to Giles and Tedds’s book and their results for Canada, similar methods are being adopted by other researchers and applied to other countries. These other applications will become the subject of scrutiny in future research, as sufficient details of their inner workings become available. In part the criticism falls on Giles and Tedds because they are pioneers in the methods. It is also unfortunate that they are vulnerable because they document their calculations and provide their data for others to analyze. In this openness and accountability they set a high standard for scientific behaviour in economics that is not always followed.
APPENDIX

TABLE A1  Variable Names and Descriptions

Indicator variables:
  CURR  Currency (notes and coins) in circulation outside of banks
  GDP  Real gross domestic product (in 1986 dollars)

Causal variables:
  MULT  Total number of male holders of multiple jobs aged 15 or older
  SELF  Income of self-employed persons
  INC  Real annual disposable income per member of the labour force
  ERTE  Nominal Canadian-US exchange rate (Cdn$/US$)
  CORP  Ratio of total corporate tax revenue to GDP
  IND  Ratio of total indirect tax revenue to GDP
  UNEM  Unemployment rate

Currency demand, other variables:
  IRATE  Bank of Canada bank rate, end of year
  NGDP  Nominal GDP (= GDP × IPD)
  IPD  Implicit GDP deflator, base = 1 in 1986


TABLE A2  Causal Variables for MIMIC Model

<table>
<thead>
<tr>
<th>Year</th>
<th>MULT</th>
<th>SELF</th>
<th>INC</th>
<th>ERTE</th>
<th>CORP</th>
<th>IND</th>
<th>UNEM</th>
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<td>1976</td>
<td>207917</td>
<td>3491691</td>
<td>25616.95</td>
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Source: Data supplied by Lindsay Tedds.
TABLE A3  Other Variables: Indicators and Currency Demand Model

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Source: Data supplied by Lindsay Tedds.

More Formal Proof of Identification Results

The estimation model 7.8 is a single-equation, nonlinear regression with additive Gaussian errors, in the generic form

$$y_t = x_t(\beta) + \varepsilon_t, \quad \varepsilon_t \sim NID(0, \sigma^2), \quad \text{for } t = 1, \ldots, N,$$

(A1)

where $x_t(\beta)$ is the regression function and $\beta$ is a vector of parameters. An elegant analytical framework for a whole host of identification and estimation issues in this class of models is the Gauss-Newton regression. Given some initial value $\tilde{\beta}$, the Gauss-Newton regression is the local linear model

$$\tilde{y}_t = x_t(\tilde{\beta}) b + \varepsilon_t,$$

(A2)

where $b = \beta - \tilde{\beta}, \tilde{y}_t = y_t - x_t(\tilde{\beta})$ and $x_t(\tilde{\beta}) = \partial x_t(\beta)/\partial \beta$ evaluated at $\tilde{\beta}$. In matrix notation for all $t = 1, \ldots, N$ observations, the initial value can be updated by the familiar least squares formula for linear regression

$$\hat{b} = (X'X)^{-1} X'\tilde{y}.$$

(A3)
The Gauss-Newton algorithm for estimating a nonlinear regression model consists of repeated application of this formula until there is no more change. In practice, well-designed software to implement the algorithm will have safeguards to assist convergence.

Davidson and MacKinnon\(^{48}\) use the Gauss-Newton regression to show that many well-known results for linear regression carry over to the more complicated setting of the nonlinear regression model. In particular, an identification breakdown in the nonlinear model is equivalent to collinearity in the Gauss-Newton regression, which will be evident from linear relationships among the columns of \(X = X(\beta)\).

Consider now the particular case of the model in 7.8. Collect the parameters in the regression function of this model into the vector

\[
\beta = \left( b_0^*, b_1, b_2, b_3, a_0, a_2 \right)'.
\]  

(A4)

The derivatives of the nonlinear regression function with respect to the parameters are then

\[
X_t(\beta) = \begin{bmatrix} 1, \log G_t, r_t, p_t, r_{t-1}, \frac{b_1 Y_{R_t}}{G_t}, \frac{b_2 Y_{R_{t-2}}}{G_t} \end{bmatrix}. \]

(A5)

where \(G_t = [(1 + a_0)Y_{R_t} + a_2 Y_{R_{t-2}}] \). I will assume that there is some variation in the growth rate of recorded real income across the sample, so that \(Y_{R_t}\) is not collinear with \(Y_{R_{t-2}}\). This assumption simply rules out a trivial identification problem; I note that the data are sufficiently variable in this way. I also assume that the parameters do not take any of the isolated values that make \(G_t\) a scaled version of \(Y_{R_t}\) (this would be a result of \(a_2 = 0\)) or of \(Y_{R_{t-2}}\) (a result of \(a_0 = -1\)). These are also ways the columns of \(X\) will be collinear, and hence \(X'X\) will be singular and the model not identified; but for the present purpose they are uninteresting special cases. I will focus on a quite general problem that pertains to any point in the parameter space of this model.

Given any values for \(b_1, a_0, \) and \(a_2\), if the sixth column of \(X\) is multiplied by \((1 + a_0)\) and added to the seventh (last) column multiplied by \(a_2\), the result is identical to the first column multiplied by \(b_1\). This indicates that these three columns are collinear. Hence the parameters that correspond to these columns, namely \(a_0, a_2, \) and \(b_0^*\) (or equivalently, \(b_0\)), cannot be separately identified. The collinearity is exact, and it holds globally throughout the parameter space.

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