Optimal Payment Strategy for Corporate Income Tax Instalments

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PRÉCIS

Cet article propose une stratégie optimale pour le paiement des acomptes provisionnels de l'impôt sur le revenu des corporations. Les résultats peuvent être utilisés dans un sens quantitatif, pour déterminer le montant des paiements proprement dits, ou dans un sens qualitatif, pour suggérer des outils possibles de planification fiscale. Le résultat qualitatif le plus surprenant est que pour une compagnie, la solution optimale consiste souvent à retarder le paiement d'acomptes provisionnels importants jusqu'à une date tardive de l'année fiscale, évitant ainsi d'accorder un prêt sans intérêt au gouvernement en effectuant un paiement en trop non intentionnel. Bien que, ce faisant, la compagnie risque de devoir payer des intérêts ainsi que des pénalités pour paiements insuffisants, cette stratégie est peu coûteuse pour l'entreprise en raison des dispositions fiscales relatives aux intérêts compensatoires.

Pour mettre en œuvre la stratégie optimale proposée ici, le gestionnaire de la compagnie doit pouvoir construire un tableau indiquant tous les montants possibles de l'impôt à payer pour l'année et les probabilités qui y sont associées, et ce dès que le premier paiement d'acomptes provisionnels arrive à échéance. Cet article expose une méthode qui utilise ces informations pour déterminer les montants optimaux de paiements ainsi que les dates optimales de paiement. La méthode peut être utilisée à l'aide de l'algorithme de programmation linéaire intégré dans Excel ou dans divers autres tableurs. Cette méthode peut ne pas être familière aux lecteurs qui n'ont pas suivi de cours universitaires de commerce ou d'économie durant les dernières années. Toutefois, étant donné qu'elle est déjà incluse dans ces tableurs d'ordinateur, ils n'ont pas besoin d'en comprendre les détails pour pouvoir l'utiliser. Ils peuvent trouver le programme informatique qui calcule les montants optimaux d'acomptes provisionnels sur le site Internet du second auteur : www.arts.uwaterloo.ca/ACCT/people/macnau.htm.

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Des extensions au modèle pour tenir compte des possibilités de transferts entre le versement des acomptes provisionnels pour l'année et d'autres comptes de Revenu Canada feront l'objet de travaux futurs. Étant donné que le modèle dans sa version actuelle ne comprend pas ces éléments, il peut produire des résultats d'une qualité quelque peu inférieure à une qualité optimale. Bien que sous-optimale, la stratégie proposée demeure néanmoins supérieure à une stratégie alternative qui consiste à effectuer un paiement excédentaire d'acomptes provisionnels accordant ainsi au gouvernement un prêt sans intérêt. Les résultats du modèle peuvent à tout le moins servir de point de départ à la planification fiscale dans la mesure où ils suggèrent des stratégies qui risquent fort d'être négligées si l'on utilise des méthodes plus conventionnelles.

ABSTRACT

This article constructs an optimal strategy for the payment of Canadian corporate income tax instalments. The results can be used in a quantitative sense, to determine actual payments, or in a qualitative sense, to suggest possible planning ideas. The most surprising qualitative result is that it is often optimal for a corporation to delay substantial amounts of instalment payments until late in the fiscal year. This reduces the risk of making an interest-free loan to the government through an unintended overpayment while costing the corporation very little. Although it would seem at first that paying instalments in this manner would subject the corporation to instalment interest and underpayment penalties, the contra-interest rules eliminate this risk.

The information requirements of the strategy are that, when the first instalment payment is due, the corporate manager can construct a table listing the possible values for the tax liability for the year and their associated probabilities. This article develops a method of determining the optimal payment amounts and timing, given this information. The method can be applied using the linear programming algorithm built into Excel or various other spreadsheet programs. This method of solution may not be familiar to readers unless they have been trained in commerce or business programs in university in recent years, but because it is already included in these computer spreadsheets they need not understand it before they can use it. The computer program that calculates optimal instalment payments is available on the second author's Web site: www.arts.uwaterloo.ca/ACCT/people/macnau.htm.

Extensions of the model to incorporate the possibility of transfers between the instalment account for the year and other Revenue Canada accounts are left to future work. Because the model does not yet incorporate these factors, it may produce somewhat suboptimal results, which overweighs the opportunity loss from giving the government an interest-free loan through an instalment overpayment. Nevertheless, the model's results can be used as a starting point for decision making, since they may suggest strategies that might be overlooked by more conventional approaches.

INTRODUCTION

The Income Tax Act[1] provides three alternative ways of paying monthly corporate income tax instalments: (1) pay 1⁄12 of the estimated tax liability for the current taxation year; (2) pay 1⁄12 of the tax liability for the immediately preceding taxation year; or (3) for the first 2 months, pay 1⁄12 of the tax liability for the second preceding taxation year, and then for the remaining 10 months, pay 1⁄10 of the excess, if any, of the tax liability for the immediately preceding taxation year over the total of the first two instalments.

Most planning ideas on corporate income tax instalment payments are based on only one of the possible values for the tax liability for the year. Thus, it is frequently suggested that if a corporation's tax liability has been increasing over time, the corporation should pay on the basis of the third method described above. Similarly, if its tax liability is expected to be lower in the current year than in the preceding two years, it is suggested that the corporation use the first method.[2] A different planning idea was common in the 1980s, when the consequences of underpayment were much less onerous than they are now—taxpayers would deliberately "borrow" from the government by paying zero or small instalments.[3] Deliberate underpayments are now rarely used.

The problem with the above planning ideas is that they do not consider the effects of uncertainty. The first instalment payment is due at the end of the first month of the taxation year even though the required amount is not known at that time (and will not be known until the year is over and the tax liability for the year has been calculated).[4] In the best Canadian discussion of this issue, Brian Carr

1 RSC 1985, c. 1 (5th Supp.), as amended (herein referred to as "the Act"). Unless otherwise stated, statutory references are to the Act.

2 See, for example, Price Waterhouse, *Corporate Tax Strategy 1997-1998* (Toronto: Butterworths, 1997), at 7-24; and Scott L. Scheuermann, "Interest on Underpaid and Overpaid Amounts," in *Income Tax Enforcement, Compliance, and Administration*, 1988 Corporate Management Tax Conference (Toronto: Canadian Tax Foundation, 1988), 10:1-40. For a review of interest and penalty provisions around the world, see United States, Joint Committee on Taxation, *Study of Present-Law Penalty and Interest Provisions as Required by Section 3801 of the Internal Revenue Service Restructuring and Reform Act of 1998 (Including Provisions Relating to Corporate Tax Shelters)—Volume I* (Washington, DC: US Government Printing Office, July 22, 1999).

3 Canada, *Report of the Auditor General of Canada to the House of Commons* (Ottawa: Supply and Services, 1988), chapter 17 ("Department of National Revenue—Taxation: Tax Collection"). Similar strategies were pursued in the United States in the 1970s; see Richard C. Stark, "The Hot Interest Nightmare" (March 25, 1991), 50 *Tax Notes* 1409-17, at 1415-16. In the academic literature, see Glenn D. Feltham and Suzanne M. Paquette, "To Pay or To Delay: An Economic Analysis of the Decision To Delay and the Incidence of Delaying the Payment of Corporate Income Tax" (November 1997), 25 *Public Finance Review* 601-28.

4 Price Waterhouse, supra footnote 2, notes that important sources of this uncertainty are unexpected capital gains and changes in corporate tax rates.

and Karen Yull suggest that taxpayers should balance the cost of underpayment and the cost of overpayment:[5]

> Taxpayers who are required to make . . . instalment payments often find themselves paying either too much or too little. Overpayments are undesirable since they constitute interest-free loans to Revenue Canada; underpayments, on the other hand, result in nondeductible interest charges and possible penalties.

A US academic article goes further, suggesting that an optimal instalment payment should have the property that the marginal cost of underpayment multiplied by the probability of underpayment equals the marginal cost of overpayment multiplied by the probability of overpayment.[6]

The above comments on instalment payments under uncertainty assume, implicitly or explicitly, that there is only a single annual instalment payment. Given that the law requires monthly payments, there is also a problem of optimal timing. It is recognized in the literature that a corporation that finds itself in an underpayment position partway through the year, thus owing interest to Canada Customs and Revenue Agency (herein referred to as "Revenue Canada"), might wish to eliminate that interest by making a deliberate overpayment. Since such an overpayment eliminates non-deductible interest, the corporation is effectively earning a tax-free rate of return.[7] The more difficult problem, which has not been addressed in the literature, is whether a corporation should engage in strategic underpayments—that is, should make low payments early in the year, with a plan to make overpayments later if its tax liability for the year turns out to be on the high side.

This article formulates a theory of the optimal payment of corporate income tax instalments under uncertainty, considering both the optimal payment amount and the optimal payment timing. We assume here that corporations form beliefs about the possible amounts of corporate income tax liability for the year, and the probabilities of each amount, although we do not discuss the process by which these beliefs are formed.[8] We then combine this information about the potential

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5 Brian R. Carr and Karen Yull, "Tax on the Instalment Plan" (August 1994), 127 *CA Magazine* 35-38, at 35. As discussed below, the cost of underpayment should really be the excess of the government's interest rate over the after-tax commercial rate.

6 Jannett Highfill, Douglas Thorson, and William V. Weber, "Tax Over-Withholding as a Response to Uncertainty" (July 1998), 26 *Public Finance Review* 376-91, at 379. The intuition is that if this condition is satisfied, increasing or decreasing the instalment payment by $1 would have exactly zero effect on the corporation's loss from paying instalments. In mathematical terms, the first derivative of the loss function is being set equal to zero. A similar condition is derived in this article in the section on the optimal payment amount.

7 David Leslie and Gena Katz, "How To Avoid Paying Interest on Unpaid Instalments," *The Financial Post*, February 28, 1997.

8 Much corporation-specific information would be required. Some statistical methods that might be used are suggested in Victor M. Guerrero and J. Alan Elizondo, "Forecasting a

tax liability with information on the economic consequences of underpayments, overpayments, and different timing patterns to determine the optimal amount of instalment payments. We describe a means of solving this problem using the linear-programming algorithm provided in common spreadsheet programs such as Excel, the program for which is available on the second author's Web site.[9] This procedure assumes that the corporation is risk-neutral—that is, risk is neither sought nor avoided, and all decisions are based on expected values of returns.

This article should be useful to practitioners, not only for the computer program, but also for the general insights it gives into the problem. In addition, it should be useful to academics studying this question by providing a guide to the important factors affecting the amount of instalment payments.[10] For example, one implication of the model presented below is that among the key factors determining instalment payments are the probabilities of different amounts of tax liability, the taxpayer's cost of capital, and the interest rate the government charges on underpayments—none of which are included in a recent experiment on this issue.[11] The article should also be of interest to economic forecasters in both government and the private sector, since the predictions on optimal timing of instalment payments may suggest a new source of uncertainty in predicting corporate tax revenues, which are already notoriously difficult to predict.[12]

Cumulative Variable Using Its Partially Accumulated Data" (June 1997), 43 *Management Science* 879-89.

9 See Alan Macnaughton's Web site at www.arts.uwaterloo.ca/ACCT/people/macnau.htm.

10 As Michael Udell notes, in "The Prepayment Position Puzzle: An Analysis of Refund and Balance Due Returns: Discussion," *1991 University of Illinois Tax Research Symposium* (Urbana-Champaign: Department of Accountancy, University of Illinois, 1992), 47-52, "What I feel is missing . . . is an incentive theory about why taxpayers should choose to be in any of . . . these categories [to underpay or overpay]." Much of this literature is motivated by the documented observation that individuals who have a balance due on filing are less compliant. See Charles Christian, Sanjay Gupta, G. Weber, and Eugene Willis, "The Relation Between the Use of Tax Preparers and Taxpayers' Prepayment Position" (1994), 16 *Journal of the American Taxation Association* 17-40; Gideon Yaniv, "Tax Compliance and Advance Tax Payments: A Prospect Theory Analysis" (1999), 52 *National Tax Journal* 753-64; and Glenn Feltham and Suzanne Paquette, "The Interrelationship Between Instalment Payments and Taxpayer Compliance," paper presented at the 22d Congress of the European Accounting Association, May 1999. On corporate income tax instalment payment behaviour, see Michael L. Moore, Bert M. Steece, and Charles W. Swenson, "Some Empirical Evidence on Taxpayer Rationality" (January 1985), 60 *The Accounting Review* 18-32.

11 Benjamin Ayers, Steven Kachelmeier, and John Robinson, "Why Do People Give Interest-Free Loans to the Government? An Experimental Study of Interim Tax Payments" (Fall 1999), 21 *Journal of the American Taxation Association*, forthcoming. This experiment concerned individuals and relatively small amounts of tax payable, so non-economic considerations may be more important in that setting.

12 Ernst & Young, *Review of the Forecasting Accuracy and Methods of the Department of Finance: Executive Summary of the Final Report* (Toronto: Ernst & Young, 1994), at 3. In the

We begin by developing an expression for the corporation's loss for any given time series of instalment payments and tax liability for the year. In formulating this loss, we model provisions of the Income Tax Act and the corporation's opportunity costs and gains. In the next section we introduce uncertainty, presenting it as evolving over time. For example, in the last month of the taxation year, a corporation has better information about its liability for the year than it did in the first month of the year.[13] We also develop, for a risk-neutral corporation, the expected loss for any time pattern of instalment payments, which is the objective function to be minimized by the corporation. Following this model development, a results section shows the contrasting effects of optimal-amount and optimal-timing considerations and presents a numerical example of the solution to an instalment problem. Finally, we set out conclusions and directions for future research.

THE LOSS FROM INSTALMENTS

Readers who are unfamiliar with mathematical notation may wish first to examine the numerical example on page 83, which conveys the essence of the article.

This section develops a mathematical expression for the corporation's economic loss from any particular pattern of instalment payments. This loss is measured relative to paying only the minimum required payments; thus, if exactly the minimum payment is made each month, this loss will be zero. As explained below, positive losses may arise from paying either more or less than the minimum amounts. Uncertainty concerning the tax liability for the year means that the minimum required payments may not be known when the instalment payments have to be made; thus, a corporation may not be able to avoid having a positive economic loss from paying instalment payments regardless of what amount it chooses to pay. A later section of the article considers payment strategies that minimize the expected value of this loss.

The Income Tax Act stipulates that corporations shall make instalment payments on or before the last day of each month.[14] If the corporation's tax liability

United States, a principal cause for error in government revenue projections is wrong estimates of the timing of revenue collections based on how taxes will be paid (statement of John Wilkins, principal and national director of Tax Policy Economics, Coopers & Lybrand LLP, before the Senate Budget Committee, February 10, 1998).

13 Steven T. Limberg, "Incorporating Economic Uncertainty into the Tax Planning Curricula" (1987), 4 *Advances in Accounting* 131-49, provides an introduction to tax decision making under uncertainty. Steven Huddart, "Employee Stock Options" (1994), 18 *Journal of Accounting & Economics* 207-31, considers the evolution of uncertainty over time in connection with the optimal exercise of employee stock options.

14 See subsection 157(1). Limited exceptions to this rule are provided for credit unions and certain cooperatives in subsection 157(2), and for corporations with less than $1,000 of tax liability for the year in subsection 157(2.1). Provinces that collect their own corporate income taxes (Ontario, Quebec, and Alberta) have broadly similar instalment structures.

for the taxation year[15] is greater than the total amount of the 12 monthly instalment payments, the excess is then due at the end of the second or third month following the end of the year (hereafter, the “balance-due day”).[16] Alternatively, if the sum of the instalment payments exceeds the corporation’s tax liability for the year, the excess is refunded at some date after the balance-due day, after Revenue Canada has assessed the corporation’s tax return. In this article, the period from the first day of the corporation’s tax year to the balance-due day is referred to as the “instalment period.”

The payment amounts at these 12 dates constitute the complete set of endogenous variables (that is, variables to be optimally determined) in this article. Tax liability for the year is treated as exogenous (that is, determined outside the model).

Although Revenue Canada does not refund instalments paid by a corporation until it has assessed the return for the taxation year, its administrative practice is to allow a corporation to transfer instalments between taxation years within the same account, to another account of the taxpayer (for example, a payroll deduction account), or to the liability of a related company.[17] Such payment transfers can take place at any time between the payment date and the earlier of the required and actual filing dates of the corporation’s tax return. Corporations may make more than one transfer request a year, and they can transfer part of a payment, all of a payment, or several payments.[18] Revenue Canada gives effect to the transfer as of the original payment date; that is, Revenue Canada treats the amount as never having been remitted as an instalment and always being remitted against the account to which the amount is transferred.[19]

This model does not consider such payment transfers because the benefit derived depends on both the rules applicable to the account to which the instalment payment is being transferred and the corporation’s payment position with respect

15 The phrase “tax liability” is used to refer to the amount specified in subparagraph 157(1)(a)(i), which is the corporation’s tax payable under parts I, I.3, VI, and VI.1 of the Act.

16 Under paragraph 157(1)(b), the more favourable third-month date is restricted to certain Canadian-controlled private corporations eligible for the small business deduction in the current or preceding year, and with taxable income for an associated group of corporations under $200,000 in the preceding year.

17 There is a proposal in the 1999 budget relating to the offsetting of interest on corporate tax underpayments for one taxation year against overpayments for another taxation year. This appears to be a separate issue, since the proposal applies only to tax other than instalments. See Canada, Department of Finance, 1999 Budget, The Budget Plan 1999, February 16, 1999, at 213.

18 See Revenue Canada, *Information Circular* 81-11R3, March 26, 1993, paragraphs 35-43; and Price Waterhouse, supra footnote 2, at 13.

19 Scheuermann, supra footnote 2, at 10:20-24.

to that account.[20] Therefore, incorporating these facts would require modelling of other accounts. As well, many additional variables would be required; instead of one payment per payment date, the corporation would also have to consider transferring any or all previous payments. To the extent that a corporation uses such transfers, the model overstates the cost of overpayments.[21]

Determining the Corporation's Loss

For any given amount of tax liability for the year, a taxpayer's loss from the payment of instalments is the sum of four amounts:

$$loss = interest + opp + penalty + stub \tag{1}$$

where *interest* is the amount of interest owing under section 161 from underpaying in the instalment period ("instalment interest"[22]), *opp* is the net opportunity cost or gain through overpaying or underpaying in the instalment period, *penalty* is the penalty under section 163.1 associated with underpayment in the instalment period, and *stub* is the net loss from waiting for a refund from the balance-due day to the date the refund is paid (the "stub period"). The four amounts are developed below. Note that each is a function of the 12 monthly instalment payments.

The focal date, or comparison date, for determining each of the four amounts above is the balance-due day. That is, all losses are taken forward or discounted to that date. Effectively, the economic loss is being measured relative to a situation in which the corporation had to pay its entire tax liability for the year on that day.

Alternative focal dates could be chosen without affecting the results. To determine the value of the loss or the expected loss at any alternative date, the loss or expected loss may simply be discounted or taken forward to that alternative focal date. If the beginning of the tax year is taken as the focal date, minimizing the corporation's loss from the payment of instalments would be equivalent to minimizing the present value, as of that date, of the future instalments to be paid.[23]

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20 For example, Scheuermann, supra footnote 2, at 10:20-21, points out that any contra-interest generated by an instalment payment made in year 1 and transferred to year 2 will be eliminated for purposes of calculating interest liability under subsection 161(2) for year 1, and the payment will not start to generate contra-interest for year 2 until the first day of year 2.

21 In our model, instalment payments are like irreversible investments in the economic literature—see Avinash K. Dixit and Robert S. Pindyck, *Investment Under Uncertainty* (Princeton, NJ: Princeton University Press, 1994). As described below, there is a benefit to waiting for the uncertainty to be resolved (that is, delaying payments so as to avoid the cost of overpayments). Payment transfers make instalment payments more like reversible investments, and reduce or eliminate the benefit to waiting.

22 Revenue Canada uses the term "instalment interest" to describe this amount in *Information Circular* 81-11R3, supra footnote 18, at paragraph 10.

23 See Glenn Feltham, "The Optimal Payment of Corporate Income Tax Instalments" (PhD dissertation, School of Accountancy, University of Waterloo, 1994), at 188-95.

Instalment Liability

As stated above, the Income Tax Act stipulates that corporations must make monthly instalment payments. Where these payments are not adequate (as defined below), the corporation owes interest to Revenue Canada under section 161. To calculate this interest, we begin by defining the instalment liability (the minimum required monthly payments), and then we determine the interest owing as a function of the difference between the corporation's instalment liability and its instalment payments.

As we noted in the introduction, subsection 157(1) provides three alternative methods for calculating the instalment liability, as follows:

- method 1—pay each month an instalment of 1⁄12 of the estimated tax liability for the current taxation year;
- method 2—pay each month an amount equal to 1⁄12 of the corporation's first instalment base;[24] and
- method 3—pay in each of the first two months an amount equal to 1⁄12 of the second instalment base, and for the remaining 10 months pay an amount equal to 1⁄10 of the remainder of the first instalment base (that is, after deducting the first two instalments from the first instalment base).[25]

Subsection 161(4.1) provides that, for the purpose of calculating interest and penalties on underpayments, the corporation is liable to pay instalments according to whichever of the three methods generates the least total amount of instalments for the year, except that in the first method the estimated tax payable is replaced by the actual tax payable.[26] Thus, the privilege extended by subsection 157(1) of paying instalments based on estimated tax payable has no economic substance—the actual tax payable is used for all calculations. The total amount of instalments for the taxation year under method 1 is $12x$, where x is 1⁄12 of the corporation's tax liability for the year. Similarly, the total amount of instalments for the taxation year under method 2 is $12b_1$, where b_1 is 1⁄12 of the corporation's first instalment base for the year.

24 The first and second instalment bases, which are defined in regulation 5301, are essentially the corporation's tax liability for the immediately preceding year and the second preceding year, respectively. These instalment bases are not affected by the application of a future year's loss to reduce taxable income. Special rules apply to corporations that have amalgamated with other corporations, and to corporations with taxation years of less than 12 months.

25 Under subsection 157(3), each of these amounts may be reduced by 1⁄12 of the corporation's dividend refund for the year and certain other amounts relating to mutual fund corporations and non-resident-owned investment corporations.

26 For example, suppose the tax liability for the year is $9,999, the first instalment base is $10,000, and the second instalment base is $1. The taxpayer is forced to use method 1 even though method 3 would be preferable when the time value of money is considered.

The calculation of the total amount of instalments for the taxation year under method 3 is more complex. In each of the first two months of the year, the corporation pays an amount b_2, where b_2 is 1⁄12 of the corporation's second instalment base for the year. In each of the remaining 10 months, the corporation will pay 1⁄10 of the "amount remaining" after deducting the amount paid under the first two instalments ($2b_2$) from the first instalment base ($12b_1$). Hence, the total amount of instalments for the taxation year under method 3 is $12b_1$ if $2b_2 \leq 12b_1$, and $2b_2$ otherwise; that is, there is no amount remaining to be paid after making the first two instalment payments.

A comparison of these expressions shows that, because of the rule that the corporation is liable to pay instalments under the method that generates the least total amount of instalments for the year, the applicable method is method 1 (total payments $12x$) if $x \leq b_1$, and method 2 (total payments $12b_1$) if $x > b_1$ and $2b_2 > 12b_1$. However, if $x > b_1$ and $2b_2 \leq 12b_1$ there is an ambiguity as to whether method 2 or method 3 is to be applied, since both methods generate total instalments of $12b_1$.

There are three reasons why method 3 should apply in some portion of this ambiguous category. First, it is a rule of interpretation of statutes that, if a provision is present in law, it is assumed to have purpose.[27] If method 2 were always chosen in this category, method 3 would not apply in any circumstance; hence, by this rule of statutory interpretation, method 3 must be selected for at least some parameter values. Second, method 3 is more favourable to the taxpayer when $b_2 < b_1$ because these payments have a lower present value since amounts are paid later in the year. It is also a rule of statutory interpretation that ambiguities are to be resolved in the taxpayer's favour.[28] Third, and most important, an option permitting the taxpayer to pay the amount with the lowest present value is consistent with the object and spirit of subsection 161(4.1), which favours the taxpayer by calculating interest and penalties on the method that produces the lowest total instalment payments.

On the basis of the above arguments, method 3 is the appropriate method in the portion of the ambiguous category above where $b_2 < b_1$, and method 2 is the appropriate method otherwise. Thus, the choice of methods can be restated as follows: method 1 if $x \leq b_1$, method 2 if $x \geq b_1$ and $b_2 \geq b_1$, and method 3 if $b_2 \leq b_1 \leq x$. Hence, the corporation's instalment liability at each month-end payment date i during the taxation year is

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27 Ruth Sullivan, *Driedger on the Construction of Statutes*, 3d ed. (Toronto: Butterworths, 1994), at 159, citing *A.G. Quebec v. Carrières Ste-Thérèse Ltée* (1985), 20 DLR (4th) 602, at 608 (SCC).

28 See the discussion of *Johns-Manville Canada Inc. v. The Queen*, 85 DTC 5373, [1985] 2 CTC 111 (SCC) and *Corporation Notre-Dame de Bon-Secours v. Communaute Urbaine de Quebec et al.*, 95 DTC 5017, [1995] 1 CTC 241 (SCC) in David G. Duff, "Interpreting the Income Tax Act—Part 1: Interpretive Doctrines" (1999), vol. 47, no. 3 *Canadian Tax Journal* 464-533.

$$q_i = \begin{cases} x & \text{for all } i = 1 \text{ to } 12 \quad \text{if } x \le b_1 \\ b_1 & \text{for all } i = 1 \text{ to } 12 \quad \text{if } b_1 \le \{x, b_2\} \\ \left.\begin{cases} b_2 & \text{for all } i = 1, 2 \\ \dfrac{1}{10}(12b_1 - 2b_2) & \text{for all } i = 3 \text{ to } 12 \end{cases}\right\} & \text{if } b_2 \le b_1 \le x \end{cases} \tag{2}$$

Instalment Interest

If a corporation has paid less than the instalment liability at any payment date, the taxpayer owes interest on the unpaid amount under subsection 161(2).[29] However, by virtue of subsection 161(2.2), any interest owing from underpaying instalments in the instalment period may be offset through overpaying other instalments in the period. Thus, the taxpayer can earn "contra-interest" or "offset interest" that will reduce (in the limit, to zero) the amount of instalment interest payable.[30]

The amount set out in subsection 161(2.2) is the amount, if any, by which paragraph 161(2.2)(a) exceeds paragraph 161(2.2)(b). Paragraph 161(2.2)(a) is the amount of interest that would be payable under subsection 161(2) if no instalments were paid. Subsection 161(2) requires that the corporation pay interest on an amount that the taxpayer "failed to pay" on or before the date the amount was required to be paid. The amount that the taxpayer "failed to pay" is thus the entire instalment liability. Hence, paragraph 161(2.2)(a) may be written as the sum,

$$\sum_{i=1}^{12} q_i g_i$$

where q_i is defined in equation 2 above, and g_i is the amount of instalment interest owing by the corporation as of the balance-due day for a deficiency of $1 arising at the payment date i. The reason this expression computes this interest as of the balance-due day is that, as noted above, this is the focal date for the computation of the taxpayer's loss in equation 1 above. This interest is non-deductible for the corporation because it is not an expense incurred for the purpose of earning income.

Subsection 248(11) requires that instalment interest be compounded daily. Therefore,

$$g_i = \left(1 + \frac{G}{365}\right)^{N_i} - 1 \tag{3}$$

29 This interest is deemed to be zero if it is $25 or less—see subsection 161(2.1). This feature is not considered in the model.

30 In paragraph 11 of *Information Circular* 81-11R3, supra footnote 18, Revenue Canada calls it the "credit instalment interest offset."

where G is the prescribed rate of interest for this purpose and N_i is the number of days from instalment payment date i to the balance-due day.[31] The prescribed rate of interest for this purpose for a particular quarter is defined in regulation 4301 to be 4 percentage points plus the average rate on 90-day treasury bills for the first month of the preceding quarter.

Paragraph 161(2.2)(b) is the amount of interest that would be paid to the corporation as refund interest under subsection 164(3) if it were applied to the instalment period, no tax was payable by the corporation for the year, and the applicable rate of interest was the interest rate applying to underpayments. This amount, which is also compounded by virtue of subsection 248(11), may be written,

$$\sum_{i=1}^{12} p_i g_i$$

where p_i is the instalment payment for month i.

The amount determined under subsection 161(2.2) is therefore the amount, if any, by which paragraph 161(2.2)(a) exceeds paragraph 161(2.2)(b), or[32]

$$interest = \max\left[0, \sum_{i=1}^{12} (q_i - p_i)g_i\right] \tag{4}$$

The amount of interest from underpayment in the instalment period is therefore a function of past and present tax liability (as q is a function of x, b_1, and b_2), payments p_1 through p_{12}, and the rates g_i. Note that the words "if any" require the use of the "maximum of" (max) operator.

Opportunity Cost or Gain During the Instalment Period

The preceding section defines instalment interest, which is the first component of the corporation's loss in equation 1 above. The second element in determining a corporation's loss from the payment of instalments is the opportunity cost or gain through overpaying or underpaying in the instalment period, *opp*. Overpaying means that extra funds must be diverted from other uses; their return in that alternative use is thus an opportunity cost to the corporation. The government does not pay the taxpayer interest on overpayments during the instalment period. As discussed below, the government begins to pay refund interest no earlier than 120 days after the end of the taxation year. But underpaying implies that funds can be diverted to other uses; their alternative return may be termed an opportunity

31 For simplicity, this expression assumes that the prescribed rate does not change from quarter to quarter. The required adjustments for a changing rate are not difficult.

32 This method of calculating instalment interest is equivalent to the running balance method used by Revenue Canada in *Information Circular* 81-11R3, supra footnote 18. See Glenn Feltham, supra footnote 22, at 177-87.

gain. Thus, the corporation's opportunity cost (or, if negative, opportunity gain) arising from any monthly payment is the difference between its payment amount (p_i) and its instalment liability for that month (q_i) multiplied by the corporation's after-tax cost of capital over the period from the payment date to the balance-due day (c_i),

$$(p_i - q_i)c_i$$

Summing over the 12 monthly payments, the corporation's net opportunity cost in the instalment period is

$$opp = \sum_{i=1}^{12} (p_i - q_i)c_i \tag{5}$$

Assuming that the corporation is increasing its borrowing or decreasing its interest-bearing investments to make the tax payments,[33] the corporation's after-tax cost of capital at time i is the interest rate on that borrowing or savings (C) multiplied by one minus the corporation's marginal tax rate (t). The taxpayer's marginal tax rate enters into the calculation because interest income is generally fully taxable, and the taxpayer can in practice (although not in law) deduct interest on funds borrowed to pay instalments.[34] The interest rate is compounded daily, since loans from financial institutions typically use daily compounding. The compounding period is from the instalment payment date to the balance-due day, since the assumed focal date for determining the corporation's loss from paying instalments is the balance-due day. Given these assumptions, the corporation's after-tax cost of capital for a \$1 payment over the period from the payment date to the balance-due day is[35]

$$c_i = \left[\left(1 + \frac{C}{365}\right)^{N_i} - 1\right] \cdot (1 - t) \tag{6}$$

Penalty

If a corporation has interest owing under section 161, it may be subject to a penalty under section 163.1 for underpayment of instalments. This penalty is

33 Joseph E. Miles, "Tax Speedups and Corporate Liquidity" (July-August 1967), 45 *Harvard Business Review* 2-12, 162, presents reasons why tax instalments are likely to be financed by short-term debt.

34 As stated by Robert Couzin, "Interest Deductibility: Discussant's Remarks," in Roy D. Hogg and Jack M. Mintz, eds., *Tax Effects on the Financing of Medium and Small Public Corporations* (Kingston, Ont.: John Deutsch Institute for the Study of Economic Policy, Queen's University, 1991), 79-84, at 79, "[A]s for Revenue practise, no one asks awkward questions such as whether corporate borrowing levels under lines of credit may have increased the day a tax instalment was paid, or whether money has been borrowed to pay interest." To avoid problems, Price Waterhouse (supra footnote 2, at 12) suggests borrowing for business operating purposes and using available cash balances to pay instalments.

35 For simplicity, this expression assumes that the cost of capital does not change from month to month. The required adjustments for a changing rate are not difficult.

equal to 50 percent of the amount, if any, by which instalment interest payable in respect of instalments as of the balance-due day exceeds the greater of (1) $1,000, and (2) 25 percent of the interest that would have been payable for the year if no instalment payments had been made for the year. This penalty therefore applies only to substantial underpayments; in particular, if the interest on deficient instalments does not exceed $1,000, there is no penalty.

The penalty may be written,

$$penalty \ = \ 0.5 \cdot \ \max\left[0, \ interest - \max\left(1000, \ 0.25 \sum_{i=1}^{12} q_i g_i\right)\right] \tag{7}$$

Substituting from equation 4 into the above equation and eliminating a redundant maximization operator, this equation may be rewritten as

$$penalty \ = \ 0.5 \cdot \ \max\left[0, \left(\sum_{i=1}^{12} (q_i - p_i) g_i\right) - \max\left(1000, \ 0.25 \sum_{i=1}^{12} q_i g_i\right)\right] \tag{8}$$

Net Loss from Waiting for a Refund

It is assumed that the corporation knows its tax liability with certainty no later than the balance-due day.[36] Thus, as of that date, a corporation's total instalments paid will either be more or less than its tax liability for the year. If the corporation has paid less than its tax liability for the year, it will pay that deficient amount on that day in order that it not incur any further losses through "arrears interest" charges under subsection 161(1).[37] Any instalment interest and penalty charges incurred during the instalment period will also be paid at that time. As discussed above, although it is theoretically possible for the government's interest rate on taxes owing to be sufficiently low relative to the corporation's after-tax cost of capital that the corporation would not pay its tax liability on that date, in practice it would be rare, and therefore it is assumed in this model that this is not the case.[38] No economic loss arises from this payment since in the model the economic loss is being measured relative to a situation in which the corporation had to pay its entire tax liability for the year on that day.

Conversely, if the corporation's payments are greater than its tax liability for the year, it will receive a refund after the tax return for the year has been filed by the corporation and assessed by Revenue Canada.[39] This can take over a year in

36 In the next section of the article, it is assumed more specifically that the corporation learns its final tax liability for the year immediately before making the last instalment payment.

37 Revenue Canada uses the term arrears interest to refer to interest accrued after the end of the instalment period—see *Information Circular* 81-11R3, supra footnote 18, at paragraph 27.

38 In other words, it is assumed that $G > C(1 - t)$.

39 The refund is smaller than indicated by this formula if the corporation owes part IV tax. For part IV tax, no instalments are required and payment is due three months after the year-end.

some cases, although it is assumed in the numerical example below that the refund date is 60 days after the balance-due day. Because the corporation does not have the use of this amount for the period between the balance-due day and the refund date (the "stub" period), an opportunity cost will arise. The opportunity cost may be written as follows:

$$\frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1 + \frac{C}{365}\right)^s - 1\right] \cdot (1 - t)}{\left(1 + \frac{C}{365}\right)^s \cdot (1 - t)} \tag{9}$$

where s (for "stub") is the number of days between the balance-due day and the refund date. The numerator represents, at the refund date, the opportunity cost to the corporation of having paid an amount greater than the corporation's tax liability in the instalment period. The denominator discounts this amount to the balance-due day (which is the focal date in our model). To simplify analysis, we assume here that this refund date is known with certainty, and thus the number of days a corporation must wait beyond the balance-due day for a refund (s) is a known quantity.

As partial compensation for this opportunity cost, the corporation may receive "refund interest" at the prescribed rate. Such interest does not accrue to corporations until at least 120 days after the end of the year (paragraph 164(3)(b)), and ends when the amount is refunded, repaid, or applied. Note that interest paid to the taxpayer is taxable. Hence the gain from interest paid by the government is

$$\frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1 + \frac{G - 0.02}{365}\right)^z - 1\right] \cdot (1 - t)}{\left(1 + \frac{C}{365}\right)^s \cdot (1 - t)} \tag{10}$$

where the prescribed rate of interest for this purpose is the rate of interest charged on underpayments less 2 percentage points, and z is the number of days, if any, between the date at which interest begins to accrue to the corporation and the refund date. The numerator represents the after-tax interest payable by Revenue Canada as of the refund date, while the denominator discounts this amount to the balance-due day.

The net loss for the stub period equals the opportunity cost less the gain from refund interest. From expressions 9 and 10 above, this is

$$stub = a \cdot \max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \quad \text{where} \quad a = 1 - \frac{\left(1 + \frac{G - 0.02}{365}\right)^z}{\left(1 + \frac{C}{365}\right)^s} \tag{11}$$

Note that a is an adjustment factor relating to the difference between the government's payment of interest and the corporation's cost of funds, which is always less than one (since z is much less than s) and usually quite close to zero. To understand it intuitively, note that it would equal zero if the government paid interest at the taxpayer's cost of funds ($G - 0.02 = C$) and the government paid interest from the balance-due day to the refund date ($z = s$). In that case, there would be no stub loss at all because there would be no economic loss in waiting for a refund.

Another way in which the stub loss would be zero is if the refund date was the balance-due day. This would occur if the corporation could effectively receive its refund by transferring it to another account on which a payment of equal or greater size would otherwise have to be made, such as perhaps the instalment account for the next taxation year or the employer source withholding account. Although (as discussed above) such transfers are not generally considered in this model, it is not difficult to allow for this particular type of transfer by setting $a = 0$.

Final Expression

A new expression for the loss from instalments may now be presented. Substituting into equation 1 above the expressions for instalment interest, the net opportunity cost or gain from overpayments or underpayments, the penalty, and the stub loss (which are shown above in equations 4, 5, 8, and 11, respectively, with supporting definitions in equations 2, 3, and 6), the new definition is

$$\begin{aligned} loss = {} & \max\left[0, \sum_{i=1}^{12} (q_i - p_i)g_i\right] + \sum_{i=1}^{12} (p_i - q_i)c_i \\ & + 0.5 \max\left[0, \left(\sum_{i=1}^{12} (q_i - p_i)g_i\right) - \max\left(1000, 0.25 \sum_{i=1}^{12} q_i g_i\right)\right] \\ & + a \cdot \max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \qquad (12) \end{aligned}$$

FROM CERTAINTY TO UNCERTAINTY

A corporation that knew its tax liability for the whole year right from the time the first instalment payment was required to be made could choose a pattern of instalments payments that would minimize the loss shown in equation 12 above. Although this strategy is unrealistic, it is useful to examine it as a building block for the extension to uncertainty.

Solving this problem appears at first to be difficult mathematically because of the presence of the "max" operators. But we can easily reformulate it as a linear programming problem. Linear programming is a fast and reliable optimization algorithm that is built into many electronic spreadsheets such as Excel as well as other computer programs used in financial planning.[40] Although the linear programming method of solution may be familiar only to readers who have recently been trained in commerce or business programs in university, the fact that it is already included in these computer programs means that others need not understand it before they can use it. All they need to understand is how to rewrite the problem in a slightly different but equivalent mathematical form.

The linear programming formulation of the problem of minimizing equation 12 above is as follows, where the choice variables are I, E, Y, and p_1 through p_{12}:

$$\text{Minimize} \quad I + \sum_{i=1}^{12} (p_i - q_i)c_i + E + Y \text{ subject to}$$

$$I \geq \sum_{i=1}^{12} (q_i - p_i)g_i$$

$$E \geq 0.5\left[\left(\sum_{i=1}^{12} (q_i - p_i)g_i\right) - \max\left(1000, 0.25 \sum_{i=1}^{12} q_i g_i\right)\right]$$

$$Y \geq a \cdot \sum_{i=1}^{12} (p_i - x) \tag{13}$$

The reason this reformulation achieves the desired result is that all linear programming problems implicitly have constraints that state that none of the choice variables of the problem can be negative. Thus the choice variables I, E, and Y replace the instalment interest, penalty, and stub loss expressions, respectively. Although the right-hand side of the constraint involving E uses a max operator, this is not a difficulty for the linear programming problem since none of the elements inside the operator are choice variables.

As a corporation's tax liability for the year is generally uncertain at all but perhaps the last of the 12 monthly payment dates, the corporation's goal is not to minimize the loss for any particular value of the tax liability for the year. Instead, the goal is to minimize the expected value of the loss over all possible tax liabilities that may occur.

Formally, let each possible value of the corporation's tax liability of the year be designated as a state of nature, and, for concreteness, let there be four such states. If we designate states of nature by the superscript j, then the four possible

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40 In Excel, choose Tools, then Solver. If Solver is not one of the choices displayed, rerun the Microsoft Office setup program and choose "add option" to install the Solver add-in to Excel. For more information on Solver, see www.frontsys.com. Since Solver has a limit of 200 decision variables, readers wishing to develop more complex versions of the model may wish to purchase Excel add-in programs such as What's*Best* (see www.lindo.com).

values of the tax liability are x^j, $j = 1,2,3,4$.[41] From equation 2, each state of nature j has associated with it a series of monthly instalment liabilities $q_1{}^j$ through $q_{12}{}^j$. These monthly instalment liabilities and the monthly instalment payments (discussed below) define amounts for each state of nature for the instalment interest, penalty, and stub loss (I^j, E^j, and Y^j). Also associated with each state of nature is the corporation's subjective probability of that state occurring (d^j). The corporation's goal is to choose a pattern of instalment payments that would minimize the expected loss, which is the sum across states of nature of the probability of that state of nature multiplied by the associated loss. Thus, the corporation's problem under uncertainty is:

$$\text{Minimize} \sum_{j=1}^{4} d^j \left[I^j + \sum_{i=1}^{12} (p_i{}^j - q_i{}^j)c_i + E^j + Y^j \right] \text{ subject to}$$

$$I^j \geq \sum_{i=1}^{12} (q_i{}^j - p_i{}^j)g_i, \quad j = 1,2,3,4$$

$$E^j \geq 0.5\left[\left(\sum_{i=1}^{12} (q_i{}^j - p_i{}^j)g_i\right) - \max\left(1000,\ 0.25 \sum_{i=1}^{12} q_i{}^j g_i\right)\right], \quad j = 1,2,3,4$$

$$Y^j \geq a \cdot \sum_{i=1}^{12} (p_i{}^j - x^j), \quad j = 1,2,3,4 \tag{14}$$

To complete this model, we must specify how the uncertainty about the year's tax liability is gradually eliminated as the year progresses. Let us assume for simplicity that information arrives twice a year, although practical applications of this model would likely feature monthly or quarterly arrival of information. Assume for illustration purposes that the corporation's taxation year ends on December 31.

On the date that the first instalment payment is required to be made (which is the end of the first month of the taxation year, or January 31), all that is known is the probability that each of the four states of nature will occur. The first six monthly instalment payments, for January to June, are made on the basis of this limited information. On the first day of the third quarter, which is July 1, the corporation discovers whether the true tax liability for the year will be one of the two lower values (denoted as values 1 and 2) or one of the two higher values (denoted as values 3 and 4). Five more instalment payments (July to November) are made on the basis of this information. On the last day of the taxation year (December 31 in this case), immediately before making the last monthly instalment

41 The use of a superscript for states of nature should not be confused with the use of superscripts as exponents in the earlier section of the article defining the corporation's loss for the year from making instalment payments. The only uses of exponents in this article are equations 3, 6, 9, 10, and 11 above.

payment, the corporation learns which of the four possible values is the correct one. Thus, all uncertainty is eliminated at that time. The corporation makes the final required monthly payment, and then pays any remaining tax liability on the balance-due day.[42]

The first six instalment payments of the taxation year must be the same for all states of nature because during that period in the year the corporation has no knowledge as to which state will occur. For the next five months of the taxation year, the corporation makes one of two possible series of payments—the "*A*" series, if it is learned on July 1 that one of the two higher values of tax liability will occur, or the "*B*" series, if instead one of the lower values will occur. For the December 31 payment, it is known for certain which of the four states of nature has occurred, and thus there are four possible payments. The corporation's optimization problem is thus to choose a series of 20—that is $(6 + (5 \times 2) + 4)$—potential payments although, of course, only 12 will actually be used. All these payment equations are solved for at the time the first instalment payment has to be made (January 31).

Some notation is required for these 20 potential payments. The 6 January-June payments are p_1 through p_6; the 10 July-November payments are $p_7{}^A$ through $p_{11}{}^A$ and $p_7{}^B$ through $p_{11}{}^B$ (which series of 5 payments is actually made depends on what information was received on July 1); and the 4 December payments are $p_{12}{}^1$, $p_{12}{}^2$, $p_{12}{}^3$, and $p_{12}{}^4$ (with the payment actually made depending on what information was received on December 31). In equation 14 above, all the possible payments are referred to as $p_i{}^j$. Although this appears to suggest that there are 48 possible payments (12 months i times four states of nature j), there are actually only 20 because the first 6 payments are the same for all states of nature, and the next 5 payments are the same between states of nature 1 and 2, and again the same between states of nature 3 and 4:

$$\begin{aligned} p_i{}^j &= p_i, & j &= 1,2,3,4 \quad \text{and} \quad & i &= 1,2,3,4,5,6 \\ p_i{}^j &= p_i{}^A, & j &= 1,2 \quad \text{and} \quad & i &= 7,8,9,10,11 \\ p_i{}^j &= p_i{}^B, & j &= 3,4 \quad \text{and} \quad & i &= 7,8,9,10,11 \end{aligned} \tag{15}$$

The problem in expression 14 above thus has a total of 32 variables and 12 constraints, plus the non-negativity constraints on all the variables, which are a part of any standard linear programming problem.[43] Defining the values of the

42 Readers familiar with the literature on option pricing will recognize this formulation as a simple binomial lattice with four states of nature. For a more general treatment of this method of modelling the reduction of uncertainty over time, see, for example, Chi-fu Huang and Robert H. Litzenberger, *Foundations for Financial Economics* (New York: North Holland, 1988).

43 For convenience, the computer program instead uses 48 payment variables (12 for each of the four states of nature) and then uses equation 15 as an additional set of 28 constraints. Thus, there are 48 variables and 40 constraints.

instalment interest, penalty, and stub loss requires one variable and one constraint for each of the four states of nature, for a total of 12 variables and 12 constraints. The additional 20 variables are the 20 potential payments.

RESULTS

Optimal Payment Amount

There are two different characteristics of an optimal strategy for payment of corporate income instalments—the payment amount (in aggregate) and the payment timing. In the numerical example below, both are determined simultaneously. However, to provide some insight into the nature of the problem and its solution, this section of the article addresses the two issues separately.

Let us consider first the optimal payment amount. To abstract from timing issues, imagine that only one instalment payment for the year is required—say, at the end of the seventh month of the taxation year, which is on July 31 for a December 31 year-end. At that time, the corporation knows the two possible values for the tax liability for the year (states of nature), although it does not know which one will actually occur. To simplify further, assume that the stub loss is zero. Since it is normally a small amount, this has no major effect on the conclusions.[44]

We can now explore the implications of different possible choices of payment amounts for one particular state of nature. The corporation has a choice of paying three possible general sizes of amounts: a "low" amount, so the corporation will incur both instalment interest and a penalty under section 163.1 if that state occurs; a "medium" amount, so the corporation incurs no penalty but does incur instalment interest; and a "high" amount, so the corporation incurs neither a penalty nor instalment interest but does incur an opportunity loss from overpayment.

Now consider the effects on the loss from this particular state of nature (as shown in equation 12 above, but without the stub loss) of increasing the payment amount by $1 in each of these three payment ranges. In the low-payment case, this will decrease the instalment interest by g_7 and the penalty by 0.5 times g_7. The increased payment also has an opportunity cost to the corporation of c_7. Therefore, the marginal effect on the loss from paying instalments in that particular state of nature is $-1.5g_7 + c_7$. In the medium-payment case, increasing the payment amount by $1 is the same except that there is no reduction in the penalty (because it is already zero). Thus, the marginal effect in this case is $-g_7 + c_7$. Finally, in the high-payment case, both instalment interest and penalty are already zero so the only effect is to increase the opportunity cost. Thus, the marginal effect is simply c_7.

44 The stub loss complicates things because it creates an additional division of the payment amounts into those with and those without a stub loss.

The objective of the corporation is not to decrease the loss from any particular state of nature but to decrease the expected loss across both states of nature. Thus, it must consider the effect of a $1 increase in payment on the expected loss. This is just the marginal effect on the loss in the first state multiplied by the probability of that state plus the similar marginal effect for the second state. Although the amount paid must be the same in the two states (since, at July 31, the corporation does not know which of the two states will occur), the marginal effects in the two states need not be the same. For example, a particular payment amount could be a "high" payment for one state but a "low" payment for the other state. Thus, the corporation must balance the effects in the two states.

This is illustrated in figure 1. The top panel shows the loss from making instalment payments for two different states of nature—a "high tax liability" state and a "low tax liability" state. The slope of each loss has three different line segments, corresponding to the three possible classes of payment amounts. The slopes of each line segment are as described above. On the far left is the line segment for a low payment amount, with a slope of $-1.5g_7 + c_7$. In the middle is the segment corresponding to a medium payment amount, with a slope of $-g_7 + c_7$. Finally, on the right-hand side is the line segment for the high payment amount of c_7.

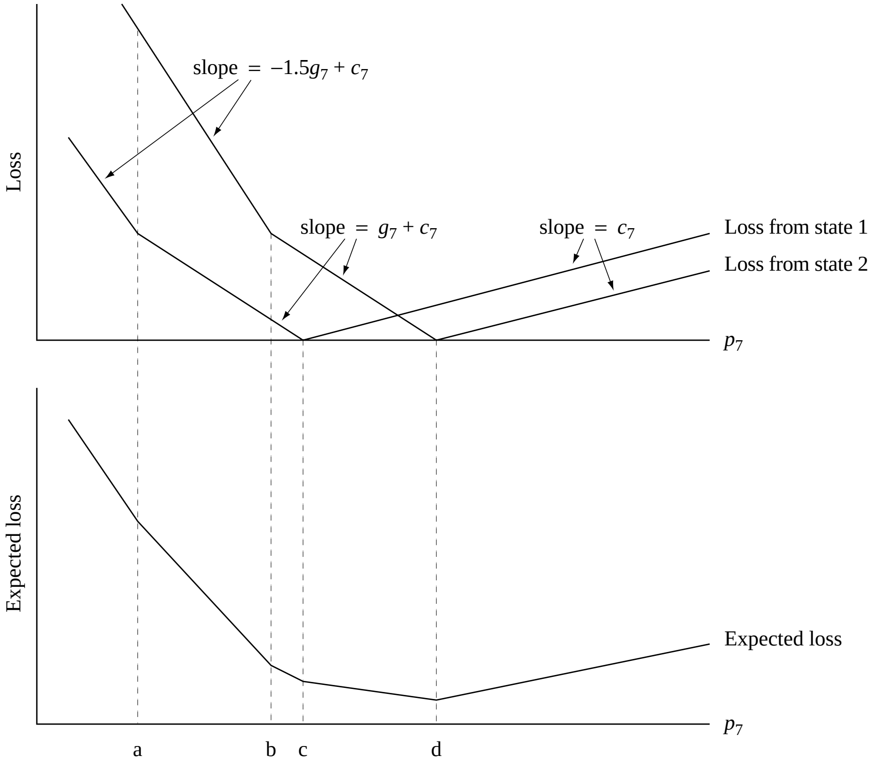
The bottom panel shows the expected loss, assuming for simplicity that both states of nature are equally probable. The minimum of this expected loss occurs at point "d," where an increase of $1 in the payment amount increases the expected loss, and a decrease in the payment amount by $1 also increases the expected loss. This minimum is likely to occur at a payment amount that corresponds to a kink in the loss from making instalments for one or other of the two states of nature (points "a" through "d" in the figure). For example, in the numerical example below, a minimum occurs at a point where, for one state of nature, the payment amount is on the borderline between being "low" and "medium"—that is, the penalty is zero but would become positive if the payment amount were reduced by even $1.

This condition for an optimal payment amount may be seen as an extension of the result from the US article that an optimal instalment payment should be such that the marginal cost of underpayment multiplied by the probability of underpayment equals the marginal cost of overpayment multiplied by the probability of overpayment.[45] Our result would simplify to essentially that rule if there were only one category of underpayment.[46] In Canada, however, there are

45 Supra footnote 6, at 379.

46 There is an additional technical difference, which is based on the fact that the US article models tax liability for the year as a continuous variable. Thus, as explained in footnote 6 above, the authors' condition for an optimum amounts to setting the first derivative of the loss function equal to zero. In contrast, we model tax liability as a discrete variable in order to

Figure 1 The Optimal Amount of a Single Instalment Payment



two categories of underpayments, since large underpayments attract a penalty while small underpayments do not.

Optimal Payment Timing

Three factors influence the optimal timing of instalment payments within the year—compound interest, the avoidance of overpayments, and the avoidance of a stub loss. The first and third factors cause a corporation to load its instalment payments early in the taxation year, while the last (and most significant) effect causes the corporation to wait and make big payments late in the year.

Consider first compound interest. Interest is compounded daily both for Revenue Canada instalment interest and penalty purposes and for computing the cost of capital (as shown in equations 3 and 6 above, respectively). That fact

allow a linear programming solution of the problem. Thus, our condition for an optimum is different because at a kink the first derivative does not exist; the derivative from the left does not equal the derivative from the right. Our condition amounts to saying that the derivative from the left must be negative or zero while the derivative from the right must be positive or zero.

implies that there is slight advantage in front-ending instalment payments. This applies even in the case of certainty.[47] Thus the solution to the problem posed in equations 12 or 13 above is not for each payment p_i to equal the monthly instalment liability q_i. Although that strategy would reduce the loss to zero, one can do better (create a negative loss) by paying just enough on the first instalment date to ensure that instalment interest for the year is zero, and then to pay the remainder on the balance-due day.

Although this effect is real, it is extremely small. In the certainty case, if the tax liability for the year and the first and second instalment bases are all \$1 million, and other parameters are those used in the numerical example below, the saving in loss from making instalments in this way is just \$46. It is useful, however, to be aware of the compound-interest factor because it explains some aspects of the results produced by the linear programming algorithm.

The second factor is avoidance of overpayments. If a corporation makes payments early in the year beyond those required for the minimum possible tax liability that the corporation expects it might have in the year, there is a chance the corporation will be in an overpayment position on the balance-due day. That is, not only might the corporation owe no instalment interest or penalty, it might also be leaving amounts on deposit with the government that are earning no interest. Formally, the taxpayer is in a position in which

$$\sum_{i=1}^{12} (p_i - q_i) g_i > 0$$

In effect, the corporation is earning contra-interest, which it cannot apply because it owes no instalment interest to apply it against. Revenue Canada does not pay such amounts to taxpayers, and therein lies the problem.

The reason that avoidance of overpayments is so important is that the contra-interest rules make it very easy to avoid underpayments. To understand this, abstract from the compound-interest issue described above by assuming that the cost of capital, instalment interest, and penalty are all compounded using simple interest with unchanging rates over the instalment period. In other words, instead of equations 3 and 6 we have

$$c_i = CN_i/365 \quad \text{and} \quad g_i = GN_i/365$$

Now consider any arbitrary series of 12 instalment payments p_1 through p_{12} and compare that to a pattern of instalment payments in which the first 11 payments are zero and the last payment is:

$$\frac{\sum_{i=1}^{12} p_i g_i}{g_{12}} \tag{16}$$

47 The reason is that the effect of compounding from paying early is greater than that from paying late. For example, if $G = 2C$, $g_i > 2c_i$.

By substituting this latter payment series in the expression for the corporation's loss from instalments in equation 12 above, we can see that the instalment interest, penalty, and net opportunity cost from overpaying or underpaying in the instalment period are exactly the same for both series of payments. In other words, apart from stub loss and the interest-compounding effect, the corporation loses nothing by waiting as long as possible to make instalment payments.[48]

The possibility of a stub loss occurs because the size of a year-end payment required to eliminate all instalment interest (equation 16 above) may be quite a bit larger than the tax liability for the year. Thus, the longer the corporation has to wait for a refund, the larger the stub loss. To arrive at an optimal timing of instalment payments, the corporation balances this effect and the interest-compounding effect against the avoidance of overpayments.

Numerical Example

In determining an optimal instalment payment strategy for a particular year, a corporation must consider both the optimal payment amount and the optimal payment timing. The following example shows how these considerations interact. Recall that a state of nature is a possible value of the tax liability for the year at the time the first instalment payment is due.

Consider a corporation with the following characteristics: first and second instalment base (b_1 and b_2) of $225,000; possible tax liabilities for the current year in the four states of nature (x^j) of zero, $100,000, $200,000, and $300,000; probabilities of the four states of nature (d^j) of 25 percent each; marginal tax rate (t) of 45 percent; pre-tax cost of capital (C) of 8 percent; prescribed rate of interest on underpayments (G) of 9 percent; balance-due day is 60 days after the end of the taxation year; estimated wait for payment of refund (s) is 60 days after the balance-due day; and the taxation year ending December 31. Some of the implications of this situation are as follows: from equation 2, monthly instalment liabilities (q_i) in the four states are zero, $8,333, $16,667, and $18,750, respectively, for every month of the year; from equation 3, the amount of instalment interest owing by the corporation for a deficiency of $1 arising at payment date i (g_i) varies from 10.20 cents for a January payment (period length 394 days) to 1.54 cents for a December payment (62 days); and from equation 6, the after-tax cost of $1 of capital from any particular payment date to the balance-due day (c_i) varies from 4.96 cents for a January payment to 0.75 cents for a December payment.

48 The model assumes that the last date to make instalment payments is the last date of the fiscal year. Actually, one can make payments right up to the balance-due day, although of course the amount of contra-interest that can be earned from any given payment amount decreases as the period between the payment and the balance-due day diminishes.

The problem chosen is challenging in that tax liability is expected to be less than both the first and second instalment bases in three of the four possible states of nature, with an aggregate probability of 75 percent. This increases the degree of uncertainty because making instalment payments based on either of method 2 or method 3 (the ones based on the past years' tax liability) is likely to result in payments that are too high.

Applying the linear programming algorithm in the Solver function in Excel to this problem yields the solution shown in table 1. The top half of the table (shaded) shows the optimal payments. Note that monthly payments grouped as shown by the dotted lines indicate situations in which states of nature cannot be distinguished and therefore a common instalment payment must be made. The bottom half presents summary information: the total payments, tax liability, balance due, and refund; the loss from paying instalments; net opportunity cost (*opp*) from overpaying or underpaying in the instalment period; instalment interest; penalty; and stub loss.

Clearly, the optimal instalment payments do not follow any of the three prescribed methods in subsection 157(1). This is because of a combination of optimal-timing and optimal-amount considerations.

Optimal-timing considerations lead to a situation in which all but 4 of the 20 possible payments are zeroes, and the positive payments are all on payment dates that immediately follow the arrival of information about possible states of nature. These critical payments are the January payment, which is the first payment of the year and reflects the existence of four states of nature; the July payment, which is just after the possible states of nature have been reduced to two; and the December payment, which is immediately after the true state of nature becomes known.[49] The rationale for this pattern is twofold: first, over a period over which information is constant, the interest-compounding factor leads the corporation to make payments as early as possible; and second, over-payment avoidance causes the corporation to delay making payments based on a high tax liability until it is certain that the high liability will indeed occur (that is, the July payments in states of nature 3 and 4 and the December payments in states of nature 2 and 4). The late payments are not so large, however, as to cause the corporation to suffer a stub loss in any state other than state 4.

Optimal amount considerations also come into play. The table shows that in three out of the four states of nature the optimal pattern of instalment payments is at a kink. In state 2, the kink concerns the penalty; the penalty is zero but would be positive if any instalment payment were reduced. In states 3 and 4, the kink concerns instalment interest; instalment interest is zero but would be positive if any instalment payment were reduced.

49 Of course, the choice of these particular payments as ones following information arrival is an arbitrary aspect of the model and is for illustration only.

Table 1 Numerical Example of Optimal Instalment Payments

	State of nature			
	State 1	State 2	State 3	State 4
	dollars			
January	32,913	32,913	32,913	32,913
February	0	0	0	0
March	0	0	0	0
April	0	0	0	0
May	0	0	0	0
June	0	0	0	0
July	0	0	153,416	153,416
August	0	0	0	0
September	0	0	0	0
October	0	0	0	0
November	0	0	0	0
December	0	67,087	0	95,021
Total	32,913	100,000	186,329	281,350
Tax liability	0	100,000	200,000	300,000
Balance due	0	0	13,671	18,650
Refund	32,913	0	0	0
Loss[a]	2,062	750	1	3
Opportunity cost	1,633	(714)	1	3
Instalment interest	0	1,464	0[b]	0[b]
Penalty	0	0[b]	0	0
Stub loss	430	0	0	0

[a] Sum of the four lines below. Totals may not add because of rounding. [b] Indicates an amount that would be positive if any payment were reduced by $1.

As noted above, the model does not consider the possibility of transferring instalments to another account of the taxpayer such as the employee source withholding account. A corporation that discovered in December that it was in state of nature 1 might wish to take advantage of this possibility since at that point an overpayment is certain, and the corporation would otherwise have to wait for a refund. Thus, the results of the model can be used as a starting point to take account transfers into consideration.

COMPARISONS WITH OTHER STRATEGIES

The benefit to be obtained from following the optimal instalment strategy shown in figure 1 can be measured by comparing it with the other commonly recommended payment strategies.

One common strategy, for which the payments are shown in table 2, is to pay the minimum required to avoid being charged instalment interest. Essentially, the corporation would be following method 2 or method 3 from subsection 157(1), which in this example both lead to payments of $18,750 per month

Table 2 Instalment Strategy That Avoids Instalment Interest

	State of nature			
	State 1	State 2	State 3	State 4
	dollars			
January	18,750	18,750	18,750	18,750
February	18,750	18,750	18,750	18,750
March	18,750	18,750	18,750	18,750
April	18,750	18,750	18,750	18,750
May	18,750	18,750	18,750	18,750
June	18,750	18,750	18,750	18,750
July	0	0	18,750	18,750
August	0	0	18,750	18,750
September	0	0	18,750	18,750
October	0	0	18,750	18,750
November	0	0	18,750	18,750
December	0	0	0	18,750
Total	112,500	112,500	206,250	225,000
Tax liability	0	100,000	200,000	300,000
Balance due	0	0	0	75,000
Refund	112,500	12,500	6,250	0
Loss[a]	5,978	1,821	653	0
Opportunity cost	4,509	1,658	572	0
Instalment interest	0	0	0	0
Penalty	0	0	0	0
Stub loss	1,470	163	82	0

[a] Sum of the four lines below. Totals may not add because of rounding.

because the first and second instalment bases are each $225,000. However, the corporation stops making instalment payments when it learns that accumulated instalments are going to exceed the tax liability for the year. This occurs on July 1 if it learns on that date that it is going to be in one of the pair of states with a low tax liability (states 1 and 2), and it occurs on December 31 if it learns on that date that it is going to be in state 3.

A second commonly recommended strategy, shown in table 3, is to use whichever of the three methods leads to the minimum monthly payment. As in the above strategy (and, coincidentally, at the same times), payments stop if the corporation learns that accumulated instalments are going to exceed the tax liability for the year. There are two further complications, both of which relate to the fact that method 1 can be chosen in this strategy. First, the tax liability for the year must be replaced by its expected value at the time the payment is to be made, since the actual tax liability is not known until the end of the year.[50]

50 This expected value is determined as follows: From January to June, it is $150,000—that is (0.25 × $0) + (0.25 × $100,000) + (0.25 × $200,000) + (0.25 × $300,000). From July to

Table 3 Instalment Strategy with the Minimum Monthly Payment

	State of nature			
	State 1	State 2	State 3	State 4
	dollars			
January	12,500	12,500	12,500	12,500
February	12,500	12,500	12,500	12,500
March	12,500	12,500	12,500	12,500
April	12,500	12,500	12,500	12,500
May	12,500	12,500	12,500	12,500
June	12,500	12,500	12,500	12,500
July	0	0	56,250	56,250
August	0	0	18,750	18,750
September	0	0	18,750	18,750
October	0	0	18,750	18,750
November	0	0	18,750	18,750
December	0	25,000	0	18,750
Total	75,000	100,000	206,250	225,000
Tax liability	0	100,000	200,000	300,000
Balance due	0	0	0	75,000
Refund	75,000	0	6,250	0
Loss[a]	3,986	343	146	539
Opportunity cost	3,006	343	64	(508)
Instalment interest	0	0	0	1,047
Penalty	0	0	0	0
Stub loss	980	0	82	0

[a] Sum of the four lines below. Totals may not add because of rounding.

Second, if method 1 has been used and further information later changes the expected value of the tax liability, a catch-up payment must be made at that later time to bring the accumulated payments to the right level—that is, in July, if the corporation learns then that it is going to be in one of the pair of states with a high tax liability (states 3 and 4), and in December if it learns then that it is going to be in state 2.

Table 4 compares the three strategies. In the example, following the optimal strategy reduces the expected loss by 67 percent relative to the strategy that avoids being charged instalment interest, and by 44 percent relative to the strategy that chooses the minimum monthly payment. The respective dollar reductions in the expected losses of $1,409 and $549 may seem small, but they

November, it is $50,000—that is (0.5 × $0) + (0.5 × $100,000)—if the corporation learns on July 1 that it is going to be in one of the pair of states with a low tax liability, and $250,000—that is (0.5 × $200,000) + (0.5 × $300,000)—otherwise. And in December it is the actual tax liability for the year, which is zero, $100,000, $200,000, and $300,000 for states 1 through 4 respectively.

Table 4 Comparison of Instalment Strategies

Strategy	Losses in each state of nature, $ State 1	State 2	State 3	State 4	Expected loss, $	Reduction in expected loss relative to optimal strategy, %
Optimal	2,062	750	1	3	704	—
Avoid instalment interest	5,978	1,821	653	0	2,113	67
Minimum payment	3,986	343	146	539	1,253	44

are driven by the assumption that the tax liability for the year varies from zero to $300,000; if the tax liabilities had varied from zero to $30 million, the figures would have been $140,900 and $54,900 instead.

Note that each of the other two strategies produces a better result in one state of nature. Thus, even though the optimal strategy is best on an ex ante basis, other strategies may do better ex post. Hence the benefits of the strategy developed in this article may be difficult to explain to clients.

CONCLUSION

This article represents a first effort to construct an optimal strategy for the payment of Canadian corporate income tax instalments. The results can be used in a quantitative sense to determine actual payments or in a qualitative sense to suggest possible planning ideas. The most surprising result is that, when there is a chance that the tax liability for the year will be less than in the previous year, it is often optimal for a corporation to delay substantial amounts of instalment payments until late in the fiscal year. This reduces the risk of making an interest-free loan to the government through an unintended overpayment while costing the corporation very little. Although it would seem at first that the corporation would be subject to instalment interest and underpayment penalty if it paid instalments in this manner, the contra-interest rules eliminate this risk.

The information requirements of the strategy are straightforward: At the time the first instalment payment is due, the corporate manager needs to be able to construct a table listing the possible values for the tax liability for the year and their associated probabilities. Given this information, the method developed in this article allows the manager to determine the optimal payment amounts and timing. The method can be applied using the linear programming algorithm built into Excel or various other spreadsheet programs.

Extensions of the model to incorporate the possibility of transfers between the instalment account for the year and other Revenue Canada accounts are left to future work. Because the model does not yet incorporate these factors, it may produce somewhat suboptimal results that overweight the opportunity loss from giving the government an interest-free loan through an instalment overpayment.

A practical implementation of this method of determining corporate income tax instalments would have to allow for the fact that partway through the year the corporation might change its opinions of the possible tax values for the year or their associated probabilities. This would not be difficult to accommodate; one would simply solve the model once more with the new information, taking past instalment payments as given. One might also want to incorporate fluctuations of interest rates by introducing uncertainty about the cost of capital or Revenue Canada's prescribed interest rate on underpayments.[51] Recently this has not been much of problem, but it has been an issue in some past years.

This model is a first step in a process to help corporations make optimal instalment payments for federal corporate income tax. Further research will extend the model to provincial corporate income tax in Ontario, Quebec, and Alberta, and to personal income tax. Meanwhile, the model's results can already be used as a starting point for decision making, since they may suggest strategies that might be overlooked by more conventional approaches.

51 Various methodologies for predicting the 90-day T-bill rate (the rate that underlies Revenue Canada's prescribed interest rate) are explored in Richard Deaves, "Forecasting Canadian Short-Term Interest Rates" (August 1996), 29 *Canadian Journal of Economics* 615-34.