
Taxes, the Discount Rate, and the Present-Value Calculation

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PRÉCIS

Cet article décrit une méthode systématique pour déterminer comment tout régime fiscal permet de : 1) modifier le taux d'actualisation; et 2) d'ajuster les bénéfices futurs dans le calcul de la valeur actualisée. Des exemples chiffrés servent à illustrer l'application de la méthode. Une forme précise d'équation différentielle est nécessaire pour que les déterminations soient effectuées correctement. L'auteur explique pourquoi la formule de la valeur actualisée permet d'obtenir la valeur marchande d'un actif. Une preuve simple du théorème d'invariance de l'impôt est fournie.

ABSTRACT

This article provides a systematic method for determining how any tax system will (1) modify the discount rate and (2) adjust the income stream in the present-value calculation. Numerical examples of the method are presented. A specific form of a differential equation is required if these determinations are to be correctly made. The reason that the present-value formula generates the market price of an asset is explained. A simple proof of the tax invariant valuations theorem is furnished.

KEYWORDS: ASSETS ■ CAPITAL GAINS ■ DEPRECIATION ■ INTEREST RATES ■ MARKET VALUE ■ VALUATION

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INTRODUCTION

Choosing a discount rate to be used in the valuation of an asset is laden with controversy. Taxation exacerbates the imbroglio by mandating adjustments to the pre-tax discount rate and by necessitating modifications to the income stream that is to be discounted. In short, the numerators and the denominators of the present-value expression are affected by a tax system. Buyers and sellers of assets need to value their assets using the present-value procedure. Those who do not treat taxes correctly in their calculations will over- or undervalue such assets.

Taxes complicate the present-value formula. Adjustments to it usually appear to be ad hoc corrections for taxes with intuitive rather than formal justifications. This article develops a consistent method for including taxes in the formula by providing the correct tax-adjusted discount rates and cash flows.

Statutory deductions for depreciation affect the price of an asset by reducing the tax outflow, and a tax on the flow of "profits" does so as well. The deduction and the tax may be applied at different rates because of the minutiae of tax law, so their effects on items in the numerators and the denominators of the present-value formula will involve terms for each type of depreciation deduction and, as will be shown, the ratios of tax factors.

Even a simple tax system requires moderately complex adjustments to the discount rate and to the income stream that is to be discounted in the present-value expression. These tax adjustments are not always obvious from an inspection of the tax system. Therefore, a systematic procedure for obtaining the tax adjustments to the present-value expression is necessary.

Unfortunately, a frequent practice is to write down a present-value formula and merely to assert that it is the one appropriate for the given tax system.¹ Dangerous assumptions may be implicitly introduced in the present-value formulas in the presence of taxes. An income tax, for example, may induce some practitioners to discount by the after-tax rate of interest, when this is not necessarily the effect of an income tax on the discount rate. Wealth taxes on market values are of special interest because they do not alter the market price of an asset; this distinction may appear implausible, but the non-recognition of it will lead to incorrect tax adjustments for the wealth tax to the present-value formula. We will show that the ad hoc method of adjusting for tax is unsafe.

The exact effects of tax systems on terms in the present-value formula are not obvious. This article develops a systematic procedure that will allow practitioners to modify the numerators and denominators of the present-value terms in ways that precisely reflect the tax system they use. Also, we shall see why the present-value formula determines the market price of an asset.

Furthermore, we shall prove that our method is correct. Technically, the integrating factor sets the precise tax adjustment to the discount rate. This tax-adjusted

1 For example, Andrew B. Lyon, "Invariant Valuation When Tax Rates Change over Time" (1990) vol. 98, no. 2 *Journal of Political Economy* 433-37.

discount rate is not necessarily the after-tax rate of return that is used so often elsewhere. The tax adjustments to the income stream to be discounted are given by converting an asset equilibrium expression into the standard form of a differential equation. These adjustments need not provide the simple after-tax cash flow.

The approach is appropriate to a wide variety of assets. It applies to real and financial assets with regular and irregular cash flows. It applies when financial instruments or other assets involve lump sums at their initial and/or terminal dates, whether or not those sums include “interest.” Capital gains or losses are explicit in this approach. It applies when taxable income is imputed, provided that the methods of imputation are clear.

Paul A. Samuelson’s three-page 1964 classic, “Tax Deductibility of Economic Depreciation To Insure Invariant Valuations,” dominates the literature in this area.² Later in this article we relate our work to his and provide a simple proof of his theorem. We adopt Samuelson’s notation and definitions. In connection with our focus, the tax papers by Alan J. Auerbach³ and Andrew B. Lyon⁴ are particularly useful in aiding the understanding of the problem addressed. Generally, this article is part of a vast literature that spreads to investment theory and beyond and is too large to cite in a meaningful way.

Whether and how an asset is financed is beyond the scope of this article. For debt and equity finance, the reader is referred to Dale W. Jorgenson and Kun-Young Yun.⁵ Note that the sinking fund method of finance involves present-value considerations.⁶ Because those funds are invested, the tax adjustments developed here apply to them.

Our study does not consider risky investments or subsequent risky cash flows, and we ignore the interaction of risk and taxation. Instead, we base our study in a riskless world, in which the interest rate is interpreted as the risk-free rate of return. The corresponding future cash flow or income streams are therefore assumed to be known with certainty. Ignoring risk limits this article, but no more so than the limitations to the use of the risk-free present-value formula itself found in the theoretical tax and valuation literature.

The section entitled “Income from an Asset and Market Return” develops the equation, excluding taxes, that practitioners require. The next section, “Taxation,” introduces taxes to that equation, and it furnishes the method for obtaining the

2 Paul A. Samuelson, “Tax Deductibility of Economic Depreciation To Insure Invariant Valuations” (1964) vol. 72, no. 6 *The Journal of Political Economy* 604-6.

3 Alan J. Auerbach, “Taxation, Corporate Financial Policy and the Cost of Capital” (1983) vol. 21, no. 3 *Journal of Economic Literature* 905-40, and “Retrospective Capital Gains Taxation” (1991) vol. 81, no. 1 *The American Economic Review* 167-78.

4 *Supra* note 1.

5 Dale W. Jorgenson and Kun-Young Yun, *Tax Reform and the Cost of Capital* (Oxford: Clarendon Press, 1991).

6 Thanks to Neil Brooks for this observation.

tax-induced modifications to the flows that are to be discounted and to the discount rate itself. The nature of these adjustments depends on the exact details of the tax system that bears on the asset. Numerical examples are provided. Following that section, “Present Value” proves the method is the correct one by deriving the present-value formula from the approach. Finally, “The Tax Invariant Valuations Theorem” provides a simplified proof of the tax invariant valuations theorem, the benchmark result in this area of tax theory.

INCOME FROM AN ASSET AND MARKET RETURN

This section establishes the ideas that are fundamental to the rest of the article, ideas that are prior to the problem of how specific tax systems require their particular modifications to the “income” terms and to the discount rate in the present-value formula. Practitioners will use these concepts to determine adjustments according to the specified tax system.

The method for choosing tax-induced modifications to the present-value formula entailed by a particular tax system will apply to any asset. That is, financial assets, rented assets, and assets dedicated to their owner’s use are all susceptible to the procedure. The “income” from the asset may include cash inflows less cash outflows. Assets reserved for their owners’ use, such as factories or machinery and owner-occupied housing, may have imputed rentals from which cash expenses are deducted. Assets for owners’ use require a special interpretation of the present-value formula (see the section on “Taxation”). Note that depreciation is not a cash outflow.

Let us use the symbol N , Samuelson’s term, to denote the net cash receipts for the asset.⁷ Deductions are assumed to be made from gross cash receipts for cash outflows attributable to owning the asset in the computation of N . Because there are no taxes in this section, there are no deductions for tax paid in calculating N at this stage, and there are no statutory depreciation allowances. The net cash receipts may vary over time; $N(t)$ denotes the flow of net cash receipts at time t . $N(t)$ is a number of dollars per unit of time. Time t may be interpreted as the present, as a past, or as a future date.

An asset has a market value at time t , which we shall indicate by $V(t)$ dollars. Asset values change over time, so we will use the symbol $V'(t)$ to indicate the time rate of change in the asset’s market price at time t . Note that the prime in $V'(t)$ distinguishes it from the capital value of the asset, $V(t)$, at time t . $V'(t)$ is a number of dollars per unit of time. If $V'(t)$ is positive, the asset is appreciating in value; if it is negative, the asset’s market value is declining at time t .

The legal and accounting meanings of capital gains and capital losses are not always clearcut, so we will clarify our usage. The economic meaning of $V'(t)$ is understandable if the asset has a market price at time t ; $V'(t)$ is, then, the change in the market price over a small period of time surrounding time t . $V'(t)$ in this sense is sometimes said to be economic depreciation. Mathematically, $V'(t)$ is precisely

7 *Supra* note 2, at 604.

defined as the derivative of $V(t)$ evaluated at time t . Following Samuelson,⁸ we will call $V'(t) > 0$ a capital gain and $V'(t) < 0$ a capital loss. The causes of the capital gain or loss are of no significance in the analysis to follow.

Asset owners are interested in the market value of the asset $V(t)$, its net cash receipts $N(t)$, and its capital gain or loss $V'(t)$. The income from the asset in the absence of tax is equal to

$$N(t) + V'(t). \quad (1)$$

This is the asset's Haig-Simons dollar income per unit of time, which is its market return if it is rented out. Haig-Simons income is the maximum amount of the income from an asset that the owners could spend on consumption without altering their wealth. Haig-Simons income includes changes in capital values (as shown in expression 1).

The market price of the asset is $V(t)$ at time t such that $V(t)$ is the amount of the owner's wealth invested in the asset. Were that amount invested elsewhere, it would earn the market rate of interest, $r(t)$, at time t , or the market return, $r(t)V(t)$.

Should the market return, $r(t)V(t)$, on that amount of capital be less than the income for the asset given by expression 1, then the asset is particularly lucrative, and its market price, $V(t)$, would be bid up. This would bring $r(t)V(t)$ into equality with the income from the asset, $N(t) + V'(t)$. But if the income for the asset given by expression 1 is below the market return on the capital—that is, if $r(t)V(t)$ exceeds $N(t) + V'(t)$ —then $V(t)$ would fall as owners withdraw their wealth from the less lucrative asset. Market forces thus bring about the equality⁹

$$r(t)V(t) = N(t) + V'(t), \quad (2)$$

where $V(t)$ is the flexible market price of the asset under consideration.

To clarify the ideas informing equation 2, consider a machine with five remaining years of useful life or a lease with five years to run from year 0 to year 4. The constant annual cash flow is to be $N(t) = \$10,000$ per year. Years remaining of income dwindle as the years pass, so the asset declines in value and there inevitably will be capital losses. If $r = 0.08$ and there are no taxes, the present value of the lease at time 0 is $V(t) = V(0) = 34,231.37$ and the capital loss will be $V'(t) = V'(0) = -7,261.49$. Thus, the left-hand side of equation 2 is

$$r(t)V(t) = rV(0) = (0.08)(34,231.37) = 2,738.51,$$

or the capital's opportunity cost. The right-hand side of equation 2 is

8 Ibid.

9 Samuelson, *ibid.*, calls equation 2 “the fundamental equation of yield.”

$$10,000 + V'(0) = 10,000 - 7,261.49 = 2,738.51,$$

or the year 0 Haig-Simons income from the asset. We have, therefore, illustrated the equality of the two sides of equation 2.¹⁰

Equation 2 is picked up again in the “Present Value” section, where we will show that the dollar amount, $V(t)$, is also equal to the present value of the net cash receipts. Equation 2 is for a no-tax world. The market rate of interest, $r(t)$, would be used in the present-value equation to calculate the price of the asset in the absence of taxes. Note that the coefficient of $V'(t)$ is unity in equation 2.

There are no taxes in equation 2, so there is no tax-deductible allowance for depreciation. Nevertheless, owners of identical assets may make their own charges for depreciation in their accounts. The owners of the identical assets may allow different amounts for depreciation, thereby arriving at different valuations for identical assets. However, those identical assets still have a single price in the market. This discrepancy highlights the irrelevance of imputed depreciation charges in a no-tax world. The net cash receipts do not contain an imputed depreciation deduction when valuing the asset in a no-tax world. We will discuss the technical reason for this in the “Present Value” section.

Equation 2, however, does contain a different sort of allowance for depreciation. The wear and tear on an asset is reflected in the capital loss or gain, represented by $V'(t)$ in equation 2. If there is no wear and tear, such as on a diamond necklace rented to a movie star, $V'(t)$ would simply reflect market conditions. If an asset deteriorates through use or with age, $V'(t)$ would be smaller but also would reflect market conditions. These properties hold whether $V'(t)$ is negative or positive.

Again, the preceding arguments assume a no-tax world, where there are no tax-deductible allowances for depreciation. The next section will show how statutory depreciation schedules affect an asset's valuation.

TAXATION

In this section, we develop a practical method for determining the adjustments to the discount rate and the net cash receipts term that are used in valuing an asset when a specified tax system applies. These adjustments must appear in the present-value expression. We will furnish an illustrative example with equation 2 as the basis of the approach. The proof of the method is in the next section.

The example below shows the effects of an income tax on the value of an asset. To remark that an income tax applies is to say very little until the tax base is defined and the tax rates applying to the components of that base are specified. We will tax the

10 These values are obtained by integrating the no-tax equation 20 below with constant $N(t) = N^*$ and constant $r(t) = r$, differentiating the solution to obtain $V'(t)$, evaluating at $t = 0$, and applying the foregoing data. The solution to present-value equation 20 in these circumstances is $V(t) = (N^*/r)(1 - e^{r(t-b)})$, $t < b$, where t and b are dates.

components of equation 2, possibly at different rates. A particular income tax system may or may not tax Haig-Simons income. We allow for either possibility by interpreting the different tax rates accordingly. If Haig-Simons income is taxed uniformly, the tax rates on all of the components of that income are equal and positive. If an income tax system omits some Haig-Simons components, the tax rates on the omitted components are zero. Note that each tax rate is expressed as a decimal.

The tax rate on the net cash receipts is τ_N . There will be a straightline depreciation allowance at rate δ on some fraction, $k \geq 0$, of historic cost, H . δ reflects an accelerated rate of depreciation as required. The value of k is another way of increasing the depreciation allowance. The taxable portion of the net cash receipts is thus $N(t) - \delta kH$ and the amount of tax on it is $\tau_N(N(t) - \delta kH)$.

The rate of capital gains tax applied to $V'(t)$ is τ_c . A negative value for $V'(t)$ would be a capital loss, which is treated as a deductible expense at rate τ_c . The amount of capital gains tax is $\tau_c V'(t)$, which may be positive or negative.

The amount of income tax is a cash outflow, so it entails deductions from the right-hand side of equation 2:

$$N(t) + V'(t) - \tau_N(N(t) - \delta kH) - \tau_c V'(t) = (1 - \tau_N)N(t) + \tau_N \delta kH + (1 - \tau_c)V'(t), \quad (3)$$

or the dollar return per unit of time on the asset after income tax.

Suppose that interest income is taxed at rate τ_r . If the capital sum, with market value $V(t)$, were invested on the open market at the after-tax market rate of interest instead of in the asset we are discussing, the amount of tax on interest income at time t would be $\tau_r r(t)V(t)$. The after-tax interest income from the alternative asset would then be

$$r(t)V(t) - \tau_r r(t)V(t) = (1 - \tau_r)r(t)V(t). \quad (4)$$

Earlier, we saw how market forces bring returns on our asset into line with the market rate of return—that is, the market price of the asset changes until it yields the market rate of return for that risk class. In the presence of taxes, it is the after-tax returns on our asset and the after-tax market returns that equilibrate. Consequently, equations 3 and 4 equalize as follows:

$$(1 - \tau_r)r(t)V(t) = (1 - \tau_N)N(t) + \tau_N \delta kH + (1 - \tau_c)V'(t). \quad (5)$$

Equation 5 is the tax counterpart of equation 2.

Let us extend the machine or lease example to the tax situation and use it as a check on equation 5. Suppose that $\tau_r = 0.25$, $\tau_N = 0.3$, $\tau_c = 0.1$ and $\delta kH = 1,000$; the tax system modifies the present value at time 0 to $V(0) = 28,478.72$; and the capital loss becomes $V'(0) = -6,212.53$. The left-hand side of equation 5 is

$$(1 - 0.25)(0.08)(28,478.72) = 1,708.723,$$

or the after-tax opportunity cost of capital. The right-hand side of equation 5 is

$$(1 - 0.3)(10,000) + 0.3(1,000) - (1 - 0.1)(6,212.53) = 1,708.723,$$

or the after-tax Haig-Simons income. Therefore, the two sides of equation 5 agree.¹¹

Equation 5 is the specification of the income tax system as we have defined it. Remember that the tax terms need not be at the same rates and that some of them may be zero, depending on the nature of the tax system. Thus, equation 5 may not necessarily specify Haig-Simons income taxation.

The market price of the asset, $V(t)$, varies so that the after-tax return to the asset equals the after-tax market return on the capital invested in it, as equation 5 requires. We will show in the next section how this $V(t)$ is also equal to the present value of the net cash receipts under this tax system. We will thereby demonstrate that market prices fluctuate to equal the tax-adjusted present value in the presence of taxes.

After one further essential manipulation, equation 5 will provide us with the adjustments to the discount rate and the net cash receipts appropriate to the specified income tax system used in the present-value formula. To arrive at the properly adjusted discount rate and the modified net cash receipts terms, we must alter equation 5 to ensure that the coefficient of $V'(t)$ is unity.

In equation 5, the specified tax system provides a coefficient of $(1 - \tau_c)$ on $V'(t)$. The division of both sides of equation 5 by $(1 - \tau_c)$ converts the coefficient of $V'(t)$ into unity—namely,

$$[(1 - \tau_r)/(1 - \tau_c)]r(t)V(t) = [1 - \tau_N]/(1 - \tau_c)N(t) + [\tau_N/(1 - \tau_c)]\delta kH + V'(t). \quad (6)$$

The division ensures that the coefficients of terms other than $V'(t)$ are ratios of tax factors with denominator $(1 - \tau_c)$.

Equation 6 gives the tax-adjusted discount rate for use in the present-value formula. This is the modified interest term associated with $V(t)$. For the tax system that we have specified, the modified discount rate is

$$[(1 - \tau_r)/(1 - \tau_c)]r(t), \quad (7)$$

or the coefficient of $V(t)$ in equation 6.¹²

Equation 6 also provides us with the terms appropriate to the given tax system that are to be put in the numerators for each period of the present-value calculation. These terms are the net cash receipts for the period adjusted in our case by the depreciation allowance and the ratio of tax factors—that is, the *first* two terms on the right-hand side of equation 6,

$$[(1 - \tau_N)/(1 - \tau_c)]N(t) + [\tau_N/(1 - \tau_c)]\delta kH. \quad (8)$$

11 These figures were obtained by solving the tax equation 15 for the constant cash flow with the tax system, obtaining $V(t) = [(\beta_2 N^* + \beta_3 \delta kH)/(\beta_1 r)](1 - e^{\beta_1 r(t-b)})$, differentiating it to obtain $V'(t)$ and applying the data.

12 $[(1 - 0.25)/(1 - 0.1)]0.8 = 0.67$.

Expression 8 is the tax-adjusted income stream that is to be discounted. Substituting the terms for the tax-adjusted discount rate from expression 7 and the tax-modified net cash receipts from expression 8, which were both obtained from equation 6, in the present-value formula gives us the price of the asset for fixed¹³ $r(t) = r$,

$$V_t = \sum_{t=1}^n \frac{[(1 - \tau_N) / (1 - \tau_c)]N_t + [\tau_N / (1 - \tau_c)]\delta kH}{[1 + (1 - \tau_r) / (1 - \tau_c)]r}^{t-1}. \quad (9)$$

This equation¹⁴ could not have been obtained by a mere inspection of the specified income tax system, which is why a systematic procedure is necessary to find the required tax adjustments to the present-value formula. In the absence of capital gains taxes, τ_c would be zero and equation 9 would revert to an equation commonly seen in corporate finance texts.

Expression 8 is the numerator in the present-value expression for time t . If net cash receipts are constant in each period, the symbol N^* would replace $N(t)$ in the formula for all periods. Notice how the ratios of tax factors appear rather than the tax factors themselves, and observe that the effect of the statutory depreciation allowance is to increase the value of the asset. It is understood in the formula that $H = 0$ for the years after the asset has been fully written off.

Under no circumstance does the capital gain or loss, $V'(t)$, appear in the present-value expression. This admonition arises from the fact that the market price, $V(t)$, is essentially the sum of the capital gains and losses, $V'(t)$, in periods prior to t . The present-value formula gives $V(t)$, so if we included $V'(t)$ in the numerator for each period as well as net cash receipts, we would be doubling up on $V(t)$.

Assets reserved for their owners' use do not have net cash receipts. To value them, an owner would have to impute the net cash receipts, $N(t)$, in equation 9. If imputed $N(t)$ exceeds market $N(t)$, the equation will give the owner's reservation price that exceeds the market price and the asset will not sell.

If there were a wealth tax at rate τ_w , the capital sum $V(t)$ would attract it no matter how it was invested. The asset we are considering and an alternative market asset would experience it. An amount $\tau_w V(t)$ would be deducted from each side of equation 5, thereby cancelling each other out. No adjustment to the present-value formula is required by a uniform wealth tax, which therefore does not affect asset values. (This answers the question posed about the treatment of a wealth tax in the introduction.) A wealth tax is not the only tax with any impact on asset values, as we shall see in "The Tax Invariant Valuations Theorem" section.

13 Variable interest rates require a product of tax factors, $(1 + br(1))(1 + br(2))^2(1 + br(3))^3 \dots (1 + br(n))^{k-1}$ in the denominator of the k th term: b is the coefficient of r in equation 9.

14 The present value from equation 9 of 32,851.27 was obtained from the earlier data using the Excel spreadsheet PV function with cash flow in advance.

PRESENT VALUE

Equation 5 is the expression for asset market equilibrium between the income components from an asset and the income from other assets, after all the specified taxes are applied. We will use equation 6 to calculate the present-value formula for the asset that yields the components of after-tax income shown on the right-hand side of equation 5. This will then be the present-value formula that is consistent with the tax system that equation 5 specifies and is consistent with portfolio equilibrium in the asset markets. Once that present-value formula has been obtained, an examination of it will provide the adjustments to the discount rate and net cash receipts terms that are defined by the tax system, which were the ones set out in the "Taxation" section.

We now rearrange equation 6,

$$V'(t) - [(1 - \tau_r)/(1 - \tau_c)]r(t)V(t) = -[(1 - \tau_N)/(1 - \tau_c)]N(t) - [\tau_N/(1 - \tau_c)]\delta kH, \quad (10)$$

define

$$\beta_1 = (1 - \tau_r)/(1 - \tau_c); \beta_2 = (1 - \tau_N)/(1 - \tau_c); \beta_3 = \tau_N/(1 - \tau_c), \quad (11)$$

and combine them in equation 10 to obtain

$$V'(t) - \beta_1 r(t)V(t) = -\beta_2 N(t) - \beta_3 \delta kH. \quad (12)$$

Equation 12 is a first-order linear differential equation with variable coefficients in standard form. The standard form requires that the asset market equilibrium equation 5 be rearranged until the coefficient of $V'(t)$ is unity, as it is in the equivalent equations 6, 10, and, most succinctly, 12. The unit coefficient on $V'(t)$ is the form essential to the solution of equation 12 using

$$e^{-\beta_1 \int r(t) dt} \quad (13)$$

as the integrating factor.¹⁵ Note especially that expression 13 has the usual form of the discount factor in a continuous-time present-value expression. It contains the tax factor β_1 , which modifies the discount rate for use in the present-value formula to

$$\beta_1 r(t) = (1 - \tau_r)/(1 - \tau_c)r(t). \quad (14)$$

The integrating factor, expression 13, is the technical method by which the tax-generated adjustments to the discount factor are found. If there were no tax, β_1 would not appear in expression 13. A different tax system will often, though not

15 William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 4th ed. (New York: Wiley, 1986), 19.

always, result in a tax-adjustment factor on the discount rate that is different from β_1 . In this way, the standard form will always provide the tax-adjusted discount rate that is appropriate to the tax system under consideration.

The solution to equation 12, by the very meaning of a solution to a differential equation, necessarily gives an expression for $V(t)$, the market price of the asset. The solution obtained by applying the integrating factor is¹⁶

$$V(t) = \int_t^s (\beta_2 N(s) + \beta_3 \delta k H) e^{-\beta_1 \int_t^s r(y) dy} ds. \quad (15)$$

Equation 15 is a present-value expression for continuous time. Its equivalent in discrete time is

$$V_t = \sum_{t=1}^n \frac{(\beta_2 N_t + \beta_3 \delta k H)}{(1 + \beta_1 r)^{t-1}}. \quad (16)$$

The equivalence of the two forms of the present-value expression can be informally intimated on a term-by-term basis for a constant interest rate, r . First, the same modified net cash receipts term is discounted in equations 15 and 16. Second, the negative sign in the exponent of e in equation 15 indicates that the whole e term is a divisor. This divisor term in e will turn out to be the denominator in equation 16. This is so because the definition of e is such that for a fixed interest rate, r ,

$$e^{\beta_1 r} = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \dots m = \frac{n}{\beta_1 r},$$

which is basically the denominator in equation 16. The relationship between the integral sign in equation 15 and the summation sign in equation 16 is that an integral is defined as the limit of a summation.

The modifications shown in present-value equations 15 and 16 mean that the technical way to find the appropriate tax-adjusted interest rate is to take the standard form of the asset market equilibrium condition that reflects the tax system, equation 12, and to examine the coefficient of $V(t)$. In our case, the coefficient is $\beta_1 r(t)$, absent sign, which is the tax-adjusted interest rate to be used in the present-value formula in equation 16. If the continuous-time approach is used, the integral including sign of that coefficient appears in the present-value equation 15, as discussed above. Also, the items to be discounted—those appearing in the numerators

16 See Boyce and DiPrima, *ibid.*, at 21-23.

of the present-value terms—are the absolute value of the right-hand side of the standard form of the asset market equilibrium equation¹⁷ $12, \beta_2 N(t) + \beta_3 \delta kH$.

Equations 15 and 16 are the equations for the present value of the tax-adjusted net cash receipts using the tax-adjusted discount rate where the particular modifications are defined by the specified tax system that define equations 5 and, ultimately, 12. Notice that the value of the asset is greater with a statutory allowance for depreciation than it would be without. This is because the statutory depreciation allowance reduces the cash outflow necessary to meet the tax liability.

This procedure also provides a systematic way of finding the tax-modified numbers that are to be discounted. As we see from equation 15 or 16, there are tax adjustments to the net cash receipts as well as a tax correction for statutory depreciation.

We see what the tax adjustments are by considering the definitions of the β s that are given in equation 11 and that appear in the present-value equations 15 and 16. Each tax adjustment, β_i , is a ratio of tax factors: one for the net cash receipts before depreciation, another for statutory depreciation, and a third for the market rate of interest.

The definition of the modifiers, the β s, used in the present-value equation are compared with the tax adjustments that were recommended in the previous section. Taking β_1 from the present-value equation 15 or 16 and comparing its definition in expression 11 with the coefficient of $r(t)$ in expression 7, we see that they are the same. Thus, the adjustment to the discount rate given by expression 7 is indeed correct. Similarly, the coefficient of $N(t)$ in equation 6 is equal to β_2 in equations 15 and 16, and the coefficient in equation 6 of δkH is equal to β_3 , by the definitions laid down in expression 11. The adjustments to the net cash receipts term and the discount rate required by the given tax system furnished in the previous section are consistent with the formula for the present value of the asset derived in this section.

It is difficult to anticipate how a given tax system will affect the discount rate and the income stream that is to be discounted to arrive at the price of an asset, which is why a systematic approach to the problem is required. The tax-adjustment factors for each of the terms in the present-value formula are ratios and the systematic procedure provided in this or the previous section is necessary to obtain them.

The changing price, $V(t)$, of this asset equilibrates the after-tax returns to this asset with after-tax returns available from other assets that may be bought in the market with that amount of capital. This is the meaning of equation 5 and is the basis for the equations in this section. By working with equation 5, we converted it into a form that gave a formula equal to the market price, $V(t)$. That formula in equation 15 or 16 is the present-value formula for the tax system. We have shown why it is that the present-value formula delivers the market price of an asset.

17 In the continuous form, we initially retain the minus signs in equation 12, which change to plus signs when manipulated into expression 13.

THE TAX INVARIANT VALUATIONS THEOREM

Samuelson¹⁸ discovered the fundamental theorem in the area of taxes and asset values. He showed, counterintuitively at the time, that depreciation is an economic concept given by the capital loss, $V'(t)$, and that the market value of an asset is unaffected by a uniform income tax on the components of its Haig-Simons income. Therefore, individuals on different rates of Haig-Simons income tax would arrive at the same valuation for a particular asset.

In our terms, uniform rates of tax on the components of Haig-Simons income mean that $\tau_r = \tau_N = \tau_c = \tau$. There is no statutory depreciation allowance, so $k = 0$, but $V'(t)$ is deductible if negative and taxable if positive. In these circumstances, equation 10 becomes

$$(1 - \tau) V'(t) - (1 - \tau)r(t)V(t) = -(1 - \tau)N(t). \quad (17)$$

Dividing by $(1 - \tau)$ —the coefficient of $V'(t)$ —to convert this differential equation to the required standard form in this case simultaneously eliminates the tax factors—that is,

$$V'(t) - r(t)V(t) = -N(t). \quad (18)$$

This standard-form differential equation is a rearranged equation 2, which was for a no-tax situation. The integrating factor for the equations is

$$e^{-\int r(y)dy}. \quad (19)$$

This gives the discount rate, $r(t)$, and the solution to equations 2 and 18—that is,

$$V(t) = \int_t^s N(s) e^{-\int_t^s r(y)dy} ds, \quad (20)$$

or the present value with a uniform Haig-Simons income tax from equation 17, and also in a no-tax world from equation 2, to prove the theorem.

We saw in the “Taxation” section that a wealth tax also does not alter asset values. Thus, a wealth tax and a uniform Haig-Simons income tax are equivalent in this sense, even if they are different rates.

CONCLUSION

Samuelson showed that uniform Haig-Simons income taxation does *not* alter asset prices. Ad hoc adjustments to the present-value formula for such a tax system will, therefore, produce the wrong valuation, which is sufficient to conclude that a

18 *Supra* note 2.

systematic procedure for modifying the present-value formula to accurately reflect a given tax system is imperative. This imperative is categorical for actual tax systems, which are always complex.

Practitioners require a method for adjusting the net cash receipts and the discount rate to accurately reflect, in the present-value formula, any particular tax system. A methodical procedure for doing so has been developed in the “Taxation” section and justified in the “Present Value” section.

The largely automatic procedure in the “Taxation” section uses an asset market equilibrium condition that links the asset price to the capital gain, the net cash receipts, and the market rate of interest. Practitioners will be able to adjust this equation to reflect the details of the tax system confronting them. The net cash receipts and the interest rate are adjusted in the way that the tax base dictates. Statutory depreciation allowances are incorporated in the method. Several income tax systems and a wealth tax have been provided as illustrations of the technique, along with a numerical example.

The adjusted equation *must* then be divided by the coefficient on the capital gains term. The net cash receipts, statutory depreciation, and the interest rate go directly into the present-value formula with their new coefficients attached to them.

The division by the coefficient of the capital gains term is essential to put the asset market equilibrium equation into a proper form. This division turns other coefficients in the equation into ratios of tax factors. Those ratios of tax factors transfer directly to the present-value equation and are the reason that ad hoc modifications to it will be unsuccessful.

The justification of this procedure in the “Present Value” section rests on the solution to the final tax-adjusted equation, which is a differential equation with an integrating factor that necessarily yields the correctly adjusted discount rate. The solution to the equation is the present value of the tax-modified net cash receipts that is mandated by a specified tax system.

These procedures, followed in the “Taxation” and “Present Value” sections, showed how asset markets make the price of an asset equal to the number of dollars returned by the present-value formula.